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AN EIGENFUNCTION PROBLEM OCCURRING IN QUANTUM MECHANICS

By E. C. TITCHMARSH (*Oxford*)

[Received 22 November 1941]

1. ACCORDING to Dirac's relativistic theory of the hydrogen atom,* the energy-levels of the atom are the eigenvalues of the linear operator

$$L \equiv -\frac{e^2}{x} - \epsilon c\left(-\frac{ih}{2\pi}\frac{d}{dx}\right) - i\epsilon\rho_3 c\frac{jh}{2\pi x} - \rho_3 mc^2.$$

Here e, c, h, and m are physical constants, j is an integer, not zero, and ϵ and ρ_3 are matrices,

$$\epsilon = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The subject of the operator is the matrix

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}.$$

The eigenvalues were obtained by Gordon.† Here I carry the analysis a stage farther, and obtain the associated expansion of an arbitrary function in terms of the eigenfunctions. I do not know whether the result is of any physical interest. It seems to be of some mathematical interest to show how it can be obtained, as the analysis differs in some respects from the ordinary 'Sturm-Liouville' examples.

The general method is similar to that which I have used in the case of the Sturm-Liouville expansion and its extensions.‡ To obtain the expansion of $\psi(x)$ we consider the solution $\psi(x,t)$ of

$$L\psi = i\frac{\partial\psi}{\partial t}$$

such that $\psi(x,0) = \psi(x)$. Let

$$\Psi_+(x,w) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty e^{iwt}\psi(x,t)\,dt, \qquad \Psi_-(x,w) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^0 e^{iwt}\psi(x,t)\,dt.$$

* P. A. M. Dirac, *Quantum Mechanics* (2nd ed.), ch. xii.

† W. Gordon, *Zeitschrift für Physik*, 48 (1928), 11–14; Dirac, loc. cit. § 74.

‡ E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 11 (1940), 129–40 and 141–6, and ibid. 12 (1941), 33–50 and 89–107.

Then $$L\Psi_+(x,w) = -\frac{i}{\sqrt{(2\pi)}}\psi(x)+w\Psi_+(x,w), \tag{1.1}$$

$$L\Psi_-(x,w) = \frac{i}{\sqrt{(2\pi)}}\psi(x)+w\Psi_-(x,w). \tag{1.2}$$

These are ordinary differential equations, which can be solved. Then, by the generalized Fourier-integral formula,

$$\psi(x,t) = \frac{1}{\sqrt{(2\pi)}}\int\limits_{ik-\infty}^{ik+\infty}\Psi_+(x,w)e^{-iwt}\,dw+\frac{1}{\sqrt{(2\pi)}}\int\limits_{-ik-\infty}^{-ik+\infty}\Psi_-(x,w)e^{-iwt}\,dw, \tag{1.3}$$

where $k > 0$. The expansion of $\psi(x)$ is obtained by taking $t = 0$ and evaluating the right-hand side by the calculus of residues. In the present case, each function is a matrix; thus

$$\Psi_+(x,w) = \begin{pmatrix}\Psi_{+,1}(x,w)\\ \Psi_{+,2}(x,w)\end{pmatrix}.$$

2. Let us write

$$\frac{2\pi e^2}{hc} = \alpha, \qquad \frac{2\pi}{hc} = \beta, \qquad \frac{2\pi mc}{h} = \gamma.$$

Then (1.1) is

$$\left(-\frac{\alpha}{x}+i\epsilon\frac{d}{dx}-i\epsilon\rho_3\frac{j}{x}-\gamma\rho_3\right)\Psi_+ = -\frac{i\beta}{\sqrt{(2\pi)}}\psi(x)+\beta w\Psi_+. \tag{2.1}$$

Since $$\epsilon\rho_3 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$

this is equivalent to the two equations

$$\left.\begin{aligned}\Psi'_{+,2}+\frac{j}{x}\Psi_{+,2}-\left(\gamma+\frac{\alpha}{x}+\beta w\right)\Psi_{+,1} &= -\frac{i\beta}{\sqrt{(2\pi)}}\psi_1(x)\\ -\Psi'_{+,1}+\frac{j}{x}\Psi_{+,1}+\left(\gamma-\frac{\alpha}{x}-\beta w\right)\Psi_{+,2} &= -\frac{i\beta}{\sqrt{(2\pi)}}\psi_2(x)\end{aligned}\right\}. \tag{2.2}$$

Consider first the corresponding homogeneous equations

$$\left.\begin{aligned}f_2'+\frac{j}{x}f_2-\left(\gamma+\frac{\alpha}{x}+\beta w\right)f_1 &= 0\\ -f_1'+\frac{j}{x}f_1+\left(\gamma-\frac{\alpha}{x}-\beta w\right)f_2 &= 0\end{aligned}\right\}. \tag{2.3}$$

For large x these approximate to

$$f_2'-(\gamma+\beta w)f_1 = 0, \qquad -f_1'+(\gamma-\beta w)f_2 = 0.$$

Putting

$$f_1(x) = \sqrt{(\gamma-\beta w)}\{P(x)-Q(x)\}, \qquad f_2(x) = \sqrt{(\gamma+\beta w)}\{P(x)+Q(x)\}$$

we obtain $\quad P'-\zeta P = 0, \qquad Q'+\zeta Q = 0,$

where $\zeta = \sqrt{(\gamma^2-\beta^2 w^2)}$, and $\mathbf{R}(\zeta) > 0$ if $\mathbf{I}(w) > 0$. These equations can now be solved. We therefore make the same substitution in (2.3), and obtain

$$\left.\begin{aligned} P'+\frac{j}{x}Q-\zeta P+\frac{\alpha\beta w}{\zeta x}P+\frac{\alpha\gamma}{\zeta x}Q = 0 \\ Q'+\frac{j}{x}P+\zeta Q-\frac{\alpha\gamma}{\zeta x}P-\frac{\alpha\beta w}{\zeta x}Q = 0 \end{aligned}\right\}. \tag{2.4}$$

Putting $P(x) = e^{-\zeta x}p(x)$, $Q(x) = e^{-\zeta x}q(x)$, we obtain

$$\left.\begin{aligned} p'+\frac{j}{x}q-2\zeta p+\frac{\alpha\beta w}{\zeta x}p+\frac{\alpha\gamma}{\zeta x}q = 0 \\ q'+\frac{j}{x}p-\frac{\alpha\gamma}{\zeta x}p-\frac{\alpha\beta w}{\zeta x}q = 0 \end{aligned}\right\}. \tag{2.5}$$

Let $\quad p(x) = x^\mu \sum_{\nu=0}^{\infty} c_\nu x^\nu, \qquad q(x) = x^\mu \sum_{\nu=0}^{\infty} d_\nu x^\nu.$

Substituting and equating coefficients of $x^{\mu+\nu-1}$, we obtain

$$\left.\begin{aligned} (\mu+\nu)c_\nu+jd_\nu-2\zeta c_{\nu-1}+\frac{\alpha\beta w}{\zeta}c_\nu+\frac{\alpha\gamma}{\zeta}d_\nu = 0 \\ (\mu+\nu)\,d_\nu+jc_\nu-\frac{\alpha\gamma}{\zeta}c_\nu-\frac{\alpha\beta w}{\zeta}d_\nu = 0 \end{aligned}\right\}. \tag{2.6}$$

For $\nu = 0$ these give

$$\left.\begin{aligned} (\mu+\alpha\beta w/\zeta)c_0+(j+\alpha\gamma/\zeta)d_0 = 0 \\ (\mu-\alpha\beta w/\zeta)d_0+(j-\alpha\gamma/\zeta)c_0 = 0 \end{aligned}\right\}. \tag{2.7}$$

These are consistent if $\mu^2 = j^2-\alpha^2$. Assuming that $\alpha < |j|$, this gives two real values for μ.

Eliminating d_ν from (2.6) we obtain

$$c_\nu = \frac{2\{(\mu+\nu)\zeta-\alpha\beta w\}}{\nu(\nu+2\mu)}c_{\nu-1}.$$

Hence $\quad p(x) = c_0 x^\mu {}_1F_1(\mu+1-\alpha\beta w/\zeta; 2\mu+1; 2\zeta x)$

in the usual hypergeometric notation. Similarly,

$$q(x) = d_0 x^\mu {}_1F_1(\mu-\alpha\beta w/\zeta; 2\mu+1; 2\zeta x).$$

The confluent hypergeometric function* $M_{k,\mu}(z)$ is defined by

$$M_{k,\mu}(z) = z^{\frac{1}{2}+\mu}e^{-\frac{1}{2}z}\,{}_1F_1(\tfrac{1}{2}+\mu-k;2\mu+1;z).$$

Thus

$$P(x) = \frac{c_0}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+\mu}} M_{\frac{\alpha\beta w}{\zeta}-\frac{1}{2},\mu}(2\zeta x), \qquad Q(x) = \frac{d_0}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+\mu}} M_{\frac{\alpha\beta w}{\zeta}+\frac{1}{2},\mu}(2\zeta x).$$

To satisfy (2.7), put

$$c_0 = A(j+\alpha\gamma/\zeta), \qquad d_0 = -A(\mu+\alpha\beta w/\zeta).$$

The general solution of (2.3) is then

$$f_{\frac{1}{2}}(x) = \frac{A\sqrt{(\gamma\mp\beta w)}}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+\mu}}\left\{\left(j+\frac{\alpha\gamma}{\zeta}\right)M_{\frac{\alpha\beta w}{\zeta}-\frac{1}{2},\mu}(2\zeta x) \pm \left(\mu+\frac{\alpha\beta w}{\zeta}\right)M_{\frac{\alpha\beta w}{\zeta}+\frac{1}{2},\mu}(2\zeta x)\right\}+$$

$$+\frac{A'\sqrt{(\gamma\mp\beta w)}}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}-\mu}}\left\{\left(j+\frac{\alpha\gamma}{\zeta}\right)M_{\frac{\alpha\beta w}{\zeta}-\frac{1}{2},-\mu}(2\zeta x) \pm \left(-\mu+\frac{\alpha\beta w}{\zeta}\right)M_{\frac{\alpha\beta w}{\zeta}+\frac{1}{2},-\mu}(2\zeta x)\right\},$$

where μ now denotes the positive value of $\sqrt{(j^2-\alpha^2)}$, and A and A' are arbitrary numbers independent of x.

We have now to select the solutions which vanish at 0 and ∞ respectively. For the former we must clearly take $A' = 0$. Taking $A = 1$,

$$f_{\frac{1}{2}}(x) = \frac{\sqrt{(\gamma\mp\beta w)}}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+\mu}}\left\{\left(j+\frac{\alpha\gamma}{\zeta}\right)M_{\frac{\alpha\beta w}{\zeta}-\frac{1}{2},\mu}(2\zeta x) \pm \left(\mu+\frac{\alpha\beta w}{\zeta}\right)M_{\frac{\alpha\beta w}{\zeta}+\frac{1}{2},\mu}(2\zeta x)\right\}. \tag{2.8}$$

To obtain the latter we observe that, if

$$W_{k,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)}M_{k,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-k)}M_{k,-\mu}(z),$$

then†

$$W_{k,\mu}(z) \sim e^{-\frac{1}{2}z}z^k \tag{2.9}$$

for large z, $|\arg z| \leqslant \pi-\delta < \pi$. Hence we take

$$A = \frac{\Gamma(-2\mu)}{\Gamma(1-\mu-\alpha\beta w/\zeta)}(2\zeta)^{\frac{1}{2}+\mu}, \qquad A' = \frac{\Gamma(2\mu)}{\Gamma(1+\mu-\alpha\beta w/\zeta)}(2\zeta)^{\frac{1}{2}-\mu}.$$

The required solutions are therefore

$$g_{\frac{1}{2}}(x) = \frac{\sqrt{(\gamma\mp\beta w)}}{x^{\frac{1}{2}}}\left\{\left(j+\frac{\alpha\gamma}{\zeta}\right)W_{\frac{\alpha\beta w}{\zeta}-\frac{1}{2},\mu}(2\zeta x) \mp W_{\frac{\alpha\beta w}{\zeta}+\frac{1}{2},\mu}(2\zeta x)\right\}. \tag{2.10}$$

* Whittaker and Watson, *Modern Analysis*, ch. xvi. † Ibid., § 16.3.

3. We have

$$\frac{d}{dx}(f_1 g_2 - f_2 g_1) = g_2\left\{\frac{j}{x}f_1+\left(\gamma-\frac{\alpha}{x}-\beta w\right)f_2\right\}-$$

$$-f_2\left\{\frac{j}{x}g_1+\left(\gamma-\frac{\alpha}{x}-\beta w\right)g_2\right\}+f_1\left\{-\frac{j}{x}g_2+\left(\gamma+\frac{\alpha}{x}+\beta w\right)g_1\right\}-$$

$$-g_1\left\{-\frac{j}{x}f_2+\left(\gamma+\frac{\alpha}{x}+\beta w\right)f_1\right\} = 0,$$

by (2.3). Hence

$$f_1 g_2 - f_2 g_1 = C, \tag{3.1}$$

where C is independent of x. Now as $x \to 0$,

$$f_{\frac{1}{2}}(x) \sim x^{\mu}\sqrt{(\gamma\mp\beta w)}\left\{\left(j+\frac{\alpha\gamma}{\zeta}\right)\pm\left(\mu+\frac{\alpha\beta w}{\zeta}\right)\right\},$$

$$g_{\frac{1}{2}}(x) \sim \frac{(2\zeta)^{\frac{1}{2}-\mu}}{x^{\mu}}\sqrt{(\gamma\mp\beta w)}\left\{\left(j+\frac{\alpha\gamma}{\zeta}\right)\mp\left(\mu-\frac{\alpha\beta w}{\zeta}\right)\right\}\frac{\Gamma(2\mu)}{\Gamma(1+\mu-\alpha\beta w/\zeta)}.$$

Hence

$$C = (2\zeta)^{\frac{1}{2}-\mu}\frac{\Gamma(2\mu+1)}{\Gamma(1+\mu-\alpha\beta w/\zeta)}\left(j+\frac{\alpha\gamma}{\zeta}\right). \tag{3.2}$$

The solution of (2.2) which vanishes at 0 and ∞ is therefore

$$\Psi_{+,\frac{1}{2}}(x,w) = -\frac{i\beta}{C\sqrt{(2\pi)}}\left[g_{\frac{1}{2}}(x)\int\limits_0^x \{f_1(y)\psi_1(y)+f_2(y)\psi_2(y)\}\,dy+\right.$$

$$\left.+f_{\frac{1}{2}}(x)\int\limits_x^{\infty}\{g_1(y)\psi_1(y)+g_2(y)\psi_2(y)\}\,dy\right]. \tag{3.3}$$

The functions $\Psi_{-,\frac{1}{2}}(x,w)$ are minus the analytic continuations of $\Psi_{+,\frac{1}{2}}(x,w)$ across the real axis between the points $w = \pm\gamma/\beta$. Hence (1.3) gives formally on putting $t = 0$

$$\psi_{\frac{1}{2}}(x) = \frac{1}{\sqrt{(2\pi)}}\left(\int\limits_{ik-\infty}^{ik+\infty} - \int\limits_{-ik-\infty}^{-ik+\infty}\right)\Psi_{+,\frac{1}{2}}(x,w)\,dw. \tag{3.4}$$

Now $\Gamma(1+\mu-\alpha\beta w/\zeta)$ has simple poles at

$$w = w_n = \frac{\gamma(\mu+n)}{\beta\sqrt{\{\alpha^2+(\mu+n)^2\}}}.$$

The residues are

$$\frac{(-1)^n\zeta_n^3}{\alpha\beta\gamma^2(n-1)!},$$

where $$\zeta_n = \sqrt{(\gamma^2-\beta^2 w_n^2)} = \frac{\alpha\gamma}{\sqrt{\{\alpha^2+(\mu+n)^2\}}}.$$

Also

$$W_{\frac{\alpha\beta w_n}{\zeta_n}\pm\frac{1}{2},\mu}(z) = W_{\mu+n\pm\frac{1}{2},\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\{\frac{1}{2}-2\mu-(n\pm\frac{1}{2})\}} M_{\mu+n\pm\frac{1}{2},\mu}(z).$$

Hence

$$g_{\frac{1}{2}}(x,w_n) = (2\zeta_n)^{\frac{1}{2}+\mu}\frac{\Gamma(-2\mu)}{\Gamma(1-n-2\mu)} f_{\frac{1}{2}}(x,w_n)$$

$$= (-1)^{n-1}(2\zeta_n)^{\frac{1}{2}+\mu}\frac{\Gamma(2\mu+n)}{\Gamma(2\mu+1)} f_{\frac{1}{2}}(x,w_n).$$

Hence the residues contribute to the right-hand side of (3.4) the series

$$S_{\frac{1}{2}} = \frac{2^{2\mu-1}\alpha^{2\mu+1}\gamma^{2\mu}}{\Gamma^2(2\mu+1)} \sum_{n=1}^{\infty} \frac{\Gamma(2\mu+n)}{(n-1)!} \frac{1}{\{\alpha^2+(\mu+n)^2\}^{\mu+1}} \frac{1}{j+\sqrt{\{\alpha^2+(\mu+n)^2\}}} \times$$

$$\times f_{\frac{1}{2}}(x,w_n)\int_0^\infty \{f_1(y,w_n)\psi_1(y)+f_2(y,w_n)\psi_2(y)\}\,dy.$$

The numbers w_n are the eigenvalues obtained by Gordon. The functions $f_{\frac{1}{2}}(x,w_n)$ are orthogonal in the sense that, if $m \neq n$,

$$\int_0^\infty \{f_1(x,w_m)f_1(x,w_n)+f_2(x,w_m)f_2(x,w_n)\}\,dx = 0,$$

as is easily verified from (2.3). They do not form a complete set since there are also integral terms in the expansion.

4. As w passes round γ/β once in the positive direction, starting and finishing at $u > \gamma/\beta$, ζ changes from $-i\sqrt{(\beta^2u^2-\gamma^2)} = -i\eta$ to $i\eta$. Hence on making $k \to 0$, the region $u > \gamma/\beta$ contributes to the right-hand side of (3.4)

$$I_{\frac{1}{2}} = \frac{i\beta}{2\pi\Gamma(2\mu+1)} \int_{\gamma/\beta}^{\infty} \frac{\Gamma(1+\mu-\alpha\beta u/i\eta)(2i\eta)^{\mu-\frac{1}{2}}}{j+\alpha\gamma/i\eta}\,du \times$$

$$\times\Big[g_{\frac{1}{2}}(x,u,i\eta)\int_0^x \{f_1(y,u,i\eta)\psi_1(y)+f_2(y,u,i\eta)\psi_2(y)\}\,dy +$$

$$+f_{\frac{1}{2}}(x,u,i\eta)\int_x^\infty \{g_1(y,u,i\eta)\psi_1(y)+g_2(y,u,i\eta)\psi_2(y)\}\,dy\Big]-$$

$$-\frac{i\beta}{2\pi\Gamma(2\mu+1)}\int_{\gamma/\beta}^{\infty}\frac{\Gamma(1+\mu+\alpha\beta u/i\eta)(-2i\eta)^{\mu-\frac{1}{2}}}{j-\alpha\gamma/i\eta}\,du\times$$

$$\times\Big[g_{\frac{1}{2}}(x,u,-i\eta)\int_{0}^{x}\{f_1(y,u,-i\eta)\psi_1(y)+f_2(y,u,-i\eta)\psi_2(y)\}\,dy+$$

$$+f_{\frac{1}{2}}(x,u,-i\eta)\int_{x}^{\infty}\{g_1(y,u,-i\eta)\psi_1(y)+g_2(y,u,-i\eta)\psi_2(y)\}\,dy\Big].$$

Now

$$f_{\frac{1}{2}}(y,u,i\eta)=\delta_{\frac{1}{2}}\frac{\sqrt{(\beta u\mp\gamma)}}{y^{\frac{1}{2}}(2i\eta)^{\frac{1}{2}+\mu}}\left\{\left(j+\frac{\alpha\gamma}{i\eta}\right)M_{\frac{\alpha\beta u}{i\eta}-\frac{1}{2},\mu}(2i\eta y)\pm\right.$$

$$\left.\pm\left(\mu+\frac{\alpha\beta u}{i\eta}\right)M_{\frac{\alpha\beta u}{i\eta}+\frac{1}{2},\mu}(2i\eta y)\right\},$$

where $\delta_1=i$, $\delta_2=1$; and $f_{\frac{1}{2}}(y,u,-i\eta)$ are the conjugates. Using the relations

$$\frac{\mu+\alpha\beta w/\zeta}{j+\alpha\gamma/\zeta}=\frac{j-\alpha\gamma/\zeta}{\mu-\alpha\beta w/\zeta} \tag{4.1}$$

and

$$z^{-\frac{1}{2}-\mu}M_{k,\mu}(z)=(-z)^{-\frac{1}{2}-\mu}M_{-k,\mu}(-z), \tag{4.2}$$

it follows that

$$f_{\frac{1}{2}}(y,u,-i\eta)=-\frac{j-\alpha\gamma/i\eta}{\mu+\alpha\beta u/i\eta}f_{\frac{1}{2}}(y,u,i\eta). \tag{4.3}$$

Hence in the above integrals we obtain $f_1(y,u,i\eta)$ multiplied by

$$\frac{\Gamma(1+\mu-\alpha\beta u/i\eta)}{j+\alpha\gamma/i\eta}(2i\eta)^{\mu-\frac{1}{2}}g_1(x,u,i\eta)+$$

$$+\Gamma(\mu+\alpha\beta u/i\eta)(-2i\eta)^{\mu-\frac{1}{2}}g_1(x,u,-i\eta). \tag{4.4}$$

Now

$$\frac{x^{\frac{1}{2}}g_1(x,u,i\eta)}{i\sqrt{(\beta u-\gamma)}}=\frac{(j+\alpha\gamma/i\eta)\Gamma(-2\mu)}{\Gamma(1-\mu-\alpha\beta u/i\eta)}M_{\frac{\alpha\beta u}{i\eta}-\frac{1}{2},\mu}(2i\eta x)+$$

$$+\frac{(j+\alpha\gamma/i\eta)\Gamma(2\mu)}{\Gamma(1+\mu-\alpha\beta u/i\eta)}M_{\frac{\alpha\beta u}{i\eta}-\frac{1}{2},-\mu}(2i\eta x)-$$

$$-\frac{\Gamma(-2\mu)}{\Gamma(-\mu-\alpha\beta u/i\eta)}M_{\frac{\alpha\beta u}{i\eta}+\frac{1}{2},\mu}(2i\eta x)-$$

$$-\frac{\Gamma(2\mu)}{\Gamma(\mu-\alpha\beta u/i\eta)}M_{\frac{\alpha\beta u}{i\eta}+\frac{1}{2},-\mu}(2i\eta x).$$

Hence by (4.2)

$$\frac{x^{\frac{1}{2}}g_1(x,u,-i\eta)}{-i\sqrt{(\beta u-\gamma)}} = \frac{(j-\alpha\gamma/i\eta)\Gamma(-2\mu)}{\Gamma(1-\mu+\alpha\beta u/i\eta)}e^{-i\pi(\frac{1}{2}+\mu)}M_{\frac{\alpha\beta u}{i\eta}+\frac{1}{2},\mu}(2i\eta x)+$$

$$+\frac{(j-\alpha\gamma/i\eta)\Gamma(2\mu)}{\Gamma(1+\mu+\alpha\beta u/i\eta)}e^{-i\pi(\frac{1}{2}-\mu)}M_{\frac{\alpha\beta u}{i\eta}+\frac{1}{2},-\mu}(2i\eta x)+$$

$$+\frac{\Gamma(-2\mu)}{\Gamma(-\mu+\alpha\beta u/i\eta)}e^{-i\pi(\frac{1}{2}+\mu)}M_{\frac{\alpha\beta u}{i\eta}-\frac{1}{2},\mu}(2i\eta x)-$$

$$-\frac{\Gamma(2\mu)}{\Gamma(\mu+\alpha\beta u/i\eta)}e^{-i\pi(\frac{1}{2}-\mu)}M_{\frac{\alpha\beta u}{i\eta}-\frac{1}{2},-\mu}(2i\eta x).$$

The coefficients of $M_{\frac{\alpha\beta u}{i\eta}\pm\frac{1}{2},-\mu}(2i\eta x)$ in (4.4) are zero. The coefficient of $M_{\frac{\alpha\beta u}{i\eta}-\frac{1}{2},\mu}(2i\eta x)$ reduces to

$$\frac{\sqrt{(\beta u-\gamma)}(2\eta)^{\mu-\frac{3}{2}}}{x^{\frac{1}{2}}\Gamma(2\mu+1)}\left(\mu-\frac{\alpha\beta u}{i\eta}\right)\left|\Gamma\left(\mu-\frac{\alpha\beta u}{i\eta}\right)\right|^2 e^{\pi\alpha\beta u/\eta}e^{-\frac{1}{2}i\pi(\mu-\frac{3}{2})}, \tag{4.5}$$

and the coefficient of $M_{\frac{\alpha\beta u}{i\eta}+\frac{1}{2},\mu}(2i\eta x)$ is

$$\left(\mu+\frac{\alpha\beta u}{i\eta}\right)\Big/\left(j+\frac{\alpha\gamma}{i\eta}\right)$$

times this expression. Hence (4.4) reduces to (4.5) multiplied by

$$M_{\frac{\alpha\beta u}{i\eta}-\frac{1}{2},\mu}(2i\eta x)+\frac{\mu+\alpha\beta u/i\eta}{j+\alpha\gamma/i\eta}M_{\frac{\alpha\beta u}{i\eta}+\frac{1}{2},\mu}(2i\eta x) = \frac{x^{\frac{1}{2}}(2i\eta)^{\frac{1}{2}+\mu}f_1(x,u,i\eta)}{i\sqrt{(\beta u-\gamma)}(j+\alpha\gamma/i\eta)}.$$

Similar analysis applies to the other terms, and we obtain finally

$$I_{\frac{1}{2}} =$$

$$-\frac{2^{2\mu-2}\beta}{\pi\Gamma^2(2\mu+1)}\int\limits_{\gamma/\beta}^{\infty}\eta^{2\mu-1}\left|\Gamma\left(\mu-\frac{\alpha\beta u}{i\eta}\right)\right|^2 e^{\pi\alpha\beta u/\eta}\frac{\mu-\alpha\beta u/i\eta}{j+\alpha\gamma/i\eta}f_{\frac{1}{2}}(x,u,i\eta)\,du\times$$

$$\times\int\limits_0^\infty\{f_1(y,u,i\eta)\psi_1(y)+f_2(y,u,i\eta)\psi_2(y)\}\,dy.$$

It follows from (4.3) that the functions

$$\left(-\frac{\mu-\alpha\beta u/i\eta}{j+\alpha\gamma/i\eta}\right)^{\frac{1}{2}}f_{\frac{1}{2}}(x,u,i\eta)$$

are real for real u and η, so that the result can be written in a purely real form. Similarly, the contribution of the region $u < -\gamma/\beta$ is

$$J_{\frac{1}{2}} = -\frac{2^{2\mu-2}\beta}{\pi\Gamma^2(2\mu+1)} \int\limits_{-\infty}^{-\gamma/\beta} \eta^{2\mu-1}\left|\Gamma\left(\mu-\frac{\alpha\beta u}{i\eta}\right)\right|^2 e^{\pi\alpha\beta u/\eta}\frac{\mu-\alpha\beta u/i\eta}{j+\alpha\gamma/i\eta} f_{\frac{1}{2}}(x,u,i\eta)\,du \times$$
$$\times \int\limits_0^\infty \{f_1(y,u,i\eta)\psi_1(y)+f_2(y,u,i\eta)\psi_2(y)\}\,dy.$$

The expansion formula is then

$$\psi_{\frac{1}{2}}(x) = S_{\frac{1}{2}}+I_{\frac{1}{2}}+J_{\frac{1}{2}}. \tag{4.6}$$

5. We shall not discuss in detail the conditions which $\psi(x)$ must satisfy for the above process to be justified, but we can state in general terms how to proceed. We have to consider the integrals of $\Psi_{+,\frac{1}{2}}(x,w)$ round a large semicircle above the real axis. Since $\alpha\beta w/\zeta \to i\alpha$ as $w \to \infty$, (2.9) gives

$$g_{\frac{1}{2}}(x,w) \sim \mp x^{-\frac{1}{2}}\sqrt{(\gamma\mp\beta w)}e^{-\zeta x}(2\zeta x)^{\frac{1}{2}+i\alpha}.$$

The asymptotic form of $M_{k,\mu}(z)$ for $\mathbf{R}(z) \to \infty$ is not stated by Whittaker and Watson, but it can be derived from the formula*

$$M_{k,\mu}(z) = \frac{\Gamma(2\mu+1)z^{\mu+\frac{1}{2}}2^{-2\mu}}{\Gamma(\frac{1}{2}+\mu+k)\Gamma(\frac{1}{2}+\mu-k)} \int\limits_{-1}^{1} (1+u)^{-\frac{1}{2}+\mu-k}(1-u)^{-\frac{1}{2}+\mu+k}e^{\frac{1}{2}zu}\,du.$$

The integral is

$$e^{\frac{1}{2}z}\int\limits_0^2 (2-v)^{-\frac{1}{2}+\mu-k}v^{-\frac{1}{2}+\mu+k}e^{-\frac{1}{2}zv}\,dv \sim e^{\frac{1}{2}z}2^{-\frac{1}{2}+\mu-k}\int\limits_0^\infty v^{-\frac{1}{2}+\mu+k}e^{-\frac{1}{2}zv}\,dv$$
$$= e^{\frac{1}{2}z}2^{-\frac{1}{2}+\mu-k}(\tfrac{1}{2}z)^{-\frac{1}{2}-\mu-k}\Gamma(\tfrac{1}{2}+\mu+k).$$

Hence $$M_{k,\mu}(z) \sim \frac{\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+\mu-k)}e^{\frac{1}{2}z}z^{-k}.$$

Thus $$f_{\frac{1}{2}}(x,w) \sim \frac{\sqrt{(\gamma\mp\beta w)}}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+\mu}}\frac{j\Gamma(2\mu+1)}{\Gamma(1+\mu-i\alpha)}(2\zeta x)^{\frac{1}{2}-i\alpha}e^{\zeta x}.$$

Hence we obtain

$$\frac{1}{C}g_{\frac{1}{2}}(x)f_{\frac{1}{2}}(y) \sim -\tfrac{1}{2}i\left(\frac{x}{y}\right)^{i\alpha}e^{i\beta u(x-y)},$$

$$\frac{1}{C}g_{\frac{1}{2}}(x)f_{\frac{2}{1}}(y) \sim \mp\tfrac{1}{2}\left(\frac{x}{y}\right)^{i\alpha}e^{i\beta u(x-y)}.$$

* Whittaker and Watson, loc. cit., Ex. 1.

For $\mathbf{I}(w)$ large and posi ive

$$\int_0^x \left(\frac{x}{y}\right)^{i\alpha} e^{i\beta w(x-y)}\psi_1(y)\,dy \sim \psi_1(x)\int_0^x e^{i\beta w(x-y)}\,dy \sim \frac{\psi_1(x)}{-i\beta w},$$

provided that ψ_1 is a sufficiently regular function; and similarly for the other terms. Hence

$$\Psi_{+,1}(x,w) \sim \frac{\psi_1(x)-i\psi_2(x)}{iw\sqrt{(2\pi)}}.$$

Thus the integral of $(2\pi)^{-\frac{1}{2}}\Psi_{+,1}(x,w)$ round a large semicircle above the real axis tends to

$$\tfrac{1}{2}\{\psi_1(x)-i\psi_2(x)\}.$$

The corresponding contribution of $(2\pi)^{-\frac{1}{2}}\Psi_{-,1}(x,w)$ is the conjugate, so that together they give $\psi_1(x)$. The total contribution of the terms involving $\psi_2(x)$ to the expansion of $\psi_1(x)$ is of course zero.

6. In the particular case $\alpha = 0$ we have

$$\begin{aligned} f_1(x) &= \frac{j\sqrt{(\gamma-\beta w)}}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+j}}\{M_{\frac{1}{2},j}(2\zeta x)+M_{-\frac{1}{2},j}(2\zeta x)\} \\ &= \frac{2j\sqrt{(\gamma-\beta w)}(2\zeta x)^{\frac{1}{2}+j}e^{-\zeta x}}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+j}}\,{}_1F_1(j;2j;2\zeta x) \\ &= 2j\sqrt{(\gamma-\beta w)}x^{\frac{1}{2}}\Gamma(j+\tfrac{1}{2})J_{j-\frac{1}{2}}(i\zeta x)/(\tfrac{1}{2}i\zeta)^{j-\frac{1}{2}}.* \end{aligned}$$

The term S_1 vanishes, and the expansion of $\psi_1(x)$ reduces to Hankel's formula involving Bessel functions of order $j-\frac{1}{2}$. Similarly

$$\begin{aligned} f_2(x) &= -\frac{j\sqrt{(\gamma+\beta w)}}{x^{\frac{1}{2}}(2\zeta)^{\frac{1}{2}+j}}\frac{(2\zeta x)^{\frac{3}{2}+j}e^{-\zeta x}}{2j+1}\,{}_1F_1(j+1;2j+1;2\zeta x) \\ &= -j\sqrt{(\gamma+\beta w)}x^{\frac{1}{2}}\Gamma(j+\tfrac{1}{2})\zeta\,J_{j+\frac{1}{2}}(i\zeta x)/(\tfrac{1}{2}i\zeta)^{j+\frac{1}{2}}, \end{aligned}$$

and the expansion of $\psi_2(x)$ reduces to Hankel's formula involving Bessel functions of order $j+\frac{1}{2}$.

* Watson, *Theory of Bessel Functions*, § 6.5 (1).

ON THE ORDER OF $\zeta(\frac{1}{2}+it)$

By E. C. TITCHMARSH (*Oxford*)

[Received 21 January 1942]

1. The object of this note is to prove that

$$\zeta(\tfrac{1}{2}+it) = O\left(t^{\frac{19}{116}}\log^{\frac{1}{58}}t\right). \tag{1.1}$$

This seems to be the best result of the kind so far obtained, the previous best, due to Phillips,* being

$$\zeta(\tfrac{1}{2}+it) = O\left(t^{\frac{229}{1392}}\right). \tag{1.2}$$

I proved (1.1) some time ago, and it has been quoted by Ingham.† The interest of such theorems, which are obviously not final, depends to a large extent on the ease with which they can be proved. I have now reduced the proof to a form which I think is not unreasonable.

The method is the two-dimensional analogue of van der Corput's method which I have applied to the problem of the lattice-points in a circle.‡ Here, however, we can replace the rather complicated estimations of integrals previously used by the following lemma.

We consider the integral

$$I = \iint\limits_D e^{2\pi i f(x,y)}\,dxdy,$$

where $f(x,y)$ is a real function with continuous derivatives of the first and second order, and D is a finite region bounded by $O(1)$ continuous monotonic arcs; and we shall suppose that all relevant curves, such as the boundary of D and the curves $f_x(x,y) = 0$, $f_y(x,y) = 0, \ldots$, intersect in $O(1)$ points. In the application these are all algebraic curves of bounded degree, so that these conditions are obviously fulfilled.

Lemma. *Let D be included in the square $|x| \leqslant R$, $|y| \leqslant R$, where $R \geqslant 2$, and let $f_{xx}(x,y) > 0$, $f_{yy}(x,y) < 0$ (or $f_{xx} < 0$, $f_{yy} > 0$), and*

$$f_{xy}(x,y) \geqslant b > 0$$

* Eric Phillips, 'The zeta-function of Riemann; further developments of van der Corput's method': *Quart. J. of Math.* (Oxford), 4 (1933), 209–25.

† A. E. Ingham, 'On the difference between consecutive primes': *Quart. J. of Math.* (Oxford), 8 (1937), 255–66.

‡ E. C. Titchmarsh, 'The lattice-points in a circle': *Proc. London Math. Soc.* (2) 38 (1934), 96–115.

throughout D. Then

$$I = O\left(\frac{\log R + |\log b|}{b}\right).$$

Consider the part D_1 of D where $f_x(x,y) > 0$, $f_y(x,y) > 0$. Let $f_y(x,y) = 0$ when $x = x_0(y)$. Then $f_y(x,y) > 0$ when $x > x_0(y)$. Let δ be a positive number not exceeding R. In the part D_1' of D_1 where $x \geqslant x_0+\delta$ (if it exists), transform to variables ξ, η, defined by $f(x,y) = \xi$, $x = \eta$. Then

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = -f_y(x,y).$$

Hence
$$\iint\limits_{D_1'} e^{2\pi i f(x,y)}\,dxdy = \iint \frac{e^{2\pi i\xi}}{f_y(x,y)}\,d\xi d\eta,$$

with appropriate limits for ξ and η. For a fixed ξ

$$\frac{\partial}{\partial\eta} f_y(x,y) = f_{xy} - \frac{f_x}{f_y} f_{yy} \geqslant b.$$

Hence
$$f_y = \int\limits_{x_0}^{\eta} \left(\frac{\partial}{\partial\eta} f_y\right) d\eta \geqslant b(\eta - x_0).$$

Also, for a fixed η, f_y is monotonic in $O(1)$ ξ-intervals, by our general conditions. Hence by the second mean-value theorem

$$\int \frac{e^{2\pi i\xi}}{f_y}\,d\xi = O\left\{\frac{1}{b(\eta - x_0)}\right\}.$$

Hence

$$\iint\limits_{D_1'} e^{2\pi i f(x,y)}\,dxdy$$

$$= O\left\{\int\limits_{x_0+\delta}^{2R} \frac{d\eta}{b(\eta - x_0)}\right\} = O\left(\frac{1}{b}\log\frac{2R - x_0}{\delta}\right) = O\left(\frac{1}{b}\log\frac{3R}{\delta}\right).$$

The integral over the remainder of D_1 is

$$O\left(\iint\limits_{x_0 \leqslant x \leqslant x_0+\delta} dxdy\right) = O(R\delta).$$

Taking $\delta = 1/(Rb)$, the result follows for D_1. The choice of δ implies that $R^2 b \geqslant 1$; we may assume this, since otherwise

$$I = O(R^2) = O(b^{-1})$$

and the result is trivial.

A similar method applies if $f_x < 0$, $f_y < 0$. If f_x and f_y have opposite signs, we put $f(x,y) = \xi$, $y = \eta$, and obtain

$$\frac{\partial}{\partial\eta} f_x = f_{xy} - \frac{f_y}{f_x} f_{xx} \geqslant b.$$

The same result therefore follows for this part. This proves the lemma.

2. We have to consider sums of the form $\sum n^{-it}$. Now the argument of § 3 of my paper* 'van der Corput's method (I)' actually shows that, if $F(n)$ is a real function, and $\rho < b-a$, then

$$\left|\sum_{n=a}^{b} e^{2\pi i F(n)}\right| \leqslant \frac{1}{\rho}\left\{4(b-a)^2\rho + 2(b-a)\left|\sum_{r=1}^{\rho-1}(\rho-r)\sum_{m=a}^{b-r} e^{2\pi i\{F(m+r)-F(m)\}}\right|\right\}^{\frac{1}{2}}. \tag{2.1}$$

Taking $F(n) = -(2\pi)^{-1}t\log n$, and using § 5 (2) of my second paper,† we have

$$\sum_{r=1}^{\rho-1}(\rho-r)\sum_{m=a}^{b-r} e^{-it\log(m+r)/m} = e^{-\frac{1}{4}\pi i}\sum_{r=1}^{\rho-1}(\rho-r)\sum_{\alpha<\nu\leqslant\beta}\frac{e^{2\pi i\phi(r,\nu)}}{|f''(m_\nu)|^{\frac{1}{2}}} +$$
$$+ O\left(a^{\frac{3}{2}}t^{-\frac{1}{2}}\rho^{\frac{3}{2}}\right) + O(\rho^2\log t) + O\left(a^{-\frac{2}{5}}t^{\frac{2}{5}}\rho^{\frac{12}{5}}\right), \tag{2.2}$$

where

$$f(x) = -\frac{t}{2\pi}\log\frac{x+r}{x}, \qquad f'(m_\nu) = \nu, \qquad \phi(\nu) = f(m_\nu) - \nu m_\nu,$$
$$\alpha = f'(b-r), \qquad \beta = f'(a), \qquad b \leqslant 2a.$$

Consider the sum

$$S = \sum_{r=R+1}^{R'}\sum_{\nu=N+1}^{N'} e^{2\pi i\phi(r,\nu)} \quad (R < R' \leqslant 2R \leqslant \rho;\ N < N' \leqslant 2N \leqslant \beta).$$

By Lemma β' of 'Lattice-points' we have, if $\lambda^4 \leqslant N'-N$,

$$S = O\left(\frac{RN}{\lambda^{\frac{1}{2}}}\right) + O\left\{\frac{R^{\frac{1}{2}}N^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\left(\sum_{\nu_1=1}^{\lambda-1}|S_1|\right)^{\frac{1}{2}}\right\}, \tag{2.3}$$

$$S_1 = O\left(\frac{RN}{\lambda}\right) + O\left\{\frac{R^{\frac{1}{2}}N^{\frac{1}{2}}}{\lambda}\left(\sum_{\nu_2=1}^{\lambda^2-1}|S_2|\right)^{\frac{1}{2}}\right\}, \tag{2.4}$$

$$S_2 = O\left(\frac{RN}{\lambda^2}\right) + O\left\{\frac{R^{\frac{1}{2}}N^{\frac{1}{2}}}{\lambda^2}\left(\sum_{\nu_3=1}^{\lambda^4-1}|S_3|\right)^{\frac{1}{2}}\right\}, \tag{2.5}$$

* E. C. Titchmarsh, 'On van der Corput's method and the zeta-function of Riemann, *Quart. J. of Math.* (Oxford), 2 (1931), 161–73.

† E. C. Titchmarsh, 'On van der Corput's method and the zeta-function of Riemann (II)': *Quart. J. of Math.* (Oxford), 2 (1931), 313–20.

where
$$S_1 = \sum_{r=R+1}^{R'} \sum_{\nu=N+1}^{N'-\nu_1} e^{2\pi i\{\phi(r,\nu+\nu_1)-\phi(r,\nu)\}}, \quad \dots$$
$$S_3 = \sum_{r=R+1}^{R'} \sum_{\nu=N+1}^{N'-\nu_1-\nu_2-\nu_3} e^{2\pi i\psi(r,\nu)},$$
$$\psi(r,\nu) = \phi(r,\nu+\nu_1+\nu_2+\nu_3) - \sum \phi(r,\nu+\nu_1+\nu_2) + \sum \phi(r,\nu+\nu_1) - \phi(r,\nu)$$
$$= \int_0^{\nu_1}\int_0^{\nu_2}\int_0^{\nu_3} \phi_{\nu\nu\nu}(r,\nu+x_1+x_2+x_3)\,dx_1\,dx_2\,dx_3.$$

Now as in § 5 of 'van der Corput's method (II)'
$$\phi_{\nu\nu}(r,\nu) = \frac{1}{2}\left(\frac{tr}{2\pi}\right)^{\frac{1}{2}}\left(\frac{1}{\nu^{\frac{3}{2}}} - \frac{\pi r}{t}\frac{1}{\nu^{\frac{1}{2}}} + \dots\right).$$

Hence
$$\phi_{rr\nu\nu\nu}(r,\nu) = \frac{1}{2}\left(\frac{t}{2\pi}\right)^{\frac{1}{2}}\left\{\frac{3}{8r^{\frac{3}{2}}\nu^{\frac{5}{2}}} + O\left(\frac{r\nu}{t}\right)\right\},$$
$$\phi_{r\nu\nu\nu\nu}(r,\nu) = \frac{1}{2}\left(\frac{t}{2\pi}\right)^{\frac{1}{2}}\left\{\frac{15}{8r^{\frac{1}{2}}\nu^{\frac{7}{2}}} + O\left(\frac{r\nu}{t}\right)\right\},$$
$$\phi_{\nu\nu\nu\nu\nu}(r,\nu) = \frac{1}{2}\left(\frac{t}{2\pi}\right)^{\frac{1}{2}}\left\{-\frac{105r^{\frac{1}{2}}}{8\nu^{\frac{9}{2}}} + O\left(\frac{r\nu}{t}\right)\right\}.$$

Divide up the region of summation of S_3 into rectangles Δ_n with sides
$$l_1 = \frac{cR^{\frac{3}{2}}N^{\frac{5}{2}}}{t^{\frac{1}{2}}\nu_1\nu_2\nu_3}, \qquad l_2 = \frac{cN^{\frac{9}{2}}}{t^{\frac{1}{2}}R^{\frac{1}{2}}\nu_1\nu_2\nu_3},$$
or parts of such rectangles. If c is small enough, ψ_r and ψ_ν vary by at most $\frac{1}{2}$ in each rectangle. Hence, by Lemma γ of 'Lattice-points', there are integers h_n, k_n such that
$$\sum_{\Delta_n}\sum e^{2\pi i\psi(r,\nu)} = \iint_{\Delta_n} e^{2\pi i\{\psi(x,y)-h_n x-k_n y\}}\,dxdy + O(l_1) + O(l_2)$$
(provided that $l_1 \geqslant 1$, $l_2 \geqslant 1$). By the above lemma, this is
$$O\left(\frac{R^{\frac{1}{2}}N^{\frac{7}{2}}}{t^{\frac{1}{2}}\nu_1\nu_2\nu_3}\log t\right) + O(l_1) + O(l_2).$$

Since
$$\nu = f'(m) = \frac{tr}{2\pi m(m+r)} > \frac{Atr}{m^2} > Ar, \tag{2.6}$$
we have $R < AN$, $l_1 < Al_2$, and hence
$$\sum_{\Delta_n}\sum e^{2\pi i\psi(r,\nu)} = O(l_2 \log t).$$

There are $O\{(Rl_1^{-1}+1)(Nl_2^{-1}+1)\}$ such rectangles. Hence

$$S_3 = O\{(RNl_1^{-1}+N+Rl_2 l_1^{-1}+l_2)\log t\}.$$

This result is also obvious if $l_1 < 1$. Hence (since $N < AN^2/R$)

$$S_3 = O\left\{\left(\frac{t^{\frac{1}{2}}\nu_1\nu_2\nu_3}{R^{\frac{1}{2}}N^{\frac{3}{2}}}+\frac{N^2}{R}+\frac{N^{\frac{9}{2}}}{t^{\frac{1}{2}}R^{\frac{1}{2}}\nu_1\nu_2\nu_3}\right)\log t\right\}.$$

Inserting this result in (2.5), then the result of that in (2.4), and so on, we obtain finally

$$S = O\left(RN\lambda^{-\frac{1}{2}}\right)+O\left(R^{\frac{13}{16}}N^{\frac{11}{16}}t^{\frac{1}{16}}\lambda^{\frac{7}{8}}\log^{\frac{1}{8}}t\right)+O\left(R^{\frac{3}{4}}N^{\frac{9}{8}}\log^{\frac{1}{8}}t\right)+$$
$$+O\left(R^{\frac{13}{16}}N^{\frac{23}{16}}t^{-\frac{1}{16}}\lambda^{-\frac{7}{8}}\log^{\frac{1}{4}}t\right). \quad (2.7)$$

The first two terms are of the same order if

$$\lambda = \left[\left(\frac{R^3N^5}{t\log^2 t}\right)^{\frac{1}{22}}\right]. \quad (2.8)$$

This gives $$S = O\left(R^{\frac{41}{44}}N^{\frac{39}{44}}t^{\frac{1}{44}}\log^{\frac{1}{22}}t\right), \quad (2.9)$$

provided that the last two terms in (2.7) are negligible. This is true if

$$N\lambda^4\log t = O(R^2), \qquad N^7\log^4 t = O(R^3t\lambda^6).$$

Using (2.8) and $N < AtRa^{-2}$ (from (2.6)), these reduce to

$$t^{19}R^5\log^7 t = O(a^{42}), \qquad t^{27}R^{10}\log^{25}t = O(a^{62}). \quad (2.10)$$

3. We now return to (2.2). We observe that the above argument applies equally well if S is taken over part of the above rectangle cut off by one or both of the curves $\nu = \alpha$, $\nu = \beta$. Also

$$|f''(x)| > At\rho a^{-3},$$

and it is easily verified that $|f''(m_\nu)|^{-\frac{1}{2}}$ satisfies the conditions of monotony necessary for partial summation in two variables. Similarly we can introduce the factor $\rho - r$ by a second partial summation. The result is

$$\sum_{r=1}^{\rho-1}(\rho-r)\sum_{\alpha<\nu\leqslant\beta}\frac{e^{2\pi i\phi(r,\nu)}}{|f''(m_\nu)|^{\frac{1}{2}}} = O\left(\rho^{\frac{63}{44}}\beta^{\frac{39}{44}}a^{\frac{3}{2}}t^{-\frac{21}{44}}\log^{\frac{1}{22}}t\right)$$
$$= O\left(\rho^{\frac{51}{22}}t^{\frac{9}{22}}a^{-\frac{3}{11}}\log^{\frac{1}{22}}t\right)$$

since $\beta = O(t\rho a^{-2})$. Hence by (2.1)

$$\sum_{n=a}^{b} n^{-it} = O\big(a\rho^{-\frac{1}{2}}\big)+O\big(a^{\frac{4}{11}}\rho^{\frac{7}{44}}t^{\frac{9}{44}}\log^{\frac{1}{44}}t\big)+O\big(a^{\frac{5}{4}}\rho^{-\frac{1}{4}}t^{-\frac{1}{4}}\big)+$$
$$+O\big(a^{\frac{1}{2}}\log^{\frac{1}{2}}t\big)+O\big(a^{\frac{3}{10}}\rho^{\frac{1}{5}}t^{\frac{1}{5}}\big).$$

The first two terms are of the same order if

$$\rho = \big[\big(a^{28}t^{-9}\log^{-1}t\big)^{\frac{1}{29}}\big], \tag{3.1}$$

and this gives

$$\sum_{n=a}^{b} n^{-it} = O\big(a^{\frac{15}{29}}t^{\frac{9}{58}}\log^{\frac{1}{58}}t\big)+O\big(a^{\frac{117}{116}}t^{-\frac{5}{29}}\log^{\frac{1}{116}}t\big)+$$
$$+O\big(a^{\frac{1}{2}}\log^{\frac{1}{2}}t\big)+O\big(a^{\frac{143}{290}}t^{\frac{4}{29}}\big). \tag{3.2}$$

The argument requires $\rho \leqslant b-a$; but the result also holds if $\rho > b-a$, since then the sum is

$$O(b-a) = O(\rho) = O\big(a^{\frac{28}{29}}t^{-\frac{9}{29}}\big),$$

which is of smaller order than the second term above.

We also notice that, if the condition $\lambda^4 \leqslant N'-N$ of § 2 is not satisfied, then

$$S = O\{R(N'-N)\} = O(R\lambda^4).$$

This is not greater than the first term on the right of (2.3) if $\lambda^9 = O(N^2)$, and it is easily verified that this is true if $a = O\big(t^{\frac{2}{3}}\big)$.

The last two terms in (3.2) are obviously negligible in comparison with the first, and the second is negligible if

$$a = O\big(t^{\frac{38}{57}-\delta}\big) \quad (\delta > 0),$$

which is true. Hence by partial summation

$$\sum_{n=a}^{b} \frac{1}{n^{\frac{1}{2}+it}} = O\big(a^{\frac{1}{58}}t^{\frac{9}{58}}\log^{\frac{1}{58}}t\big). \tag{3.3}$$

By (3.1), the conditions (2.10) now reduce to

$$t^{253}\log^{99}t = O(a^{539}), \qquad t^{231}\log^{\frac{715}{3}}t = O(a^{506}),$$

of which the former is more stringent. Hence (3.3) holds if

$$T \leqslant a < At^{\frac{1}{2}},$$

where

$$T = A_1 t^{\frac{253}{539}}\log^{\frac{99}{539}}t.$$

Adding sums of the type (3.3), we therefore obtain

$$\sum_{T\leqslant n<At^{\frac{1}{2}}} \frac{1}{n^{\frac{1}{2}+it}} = O\big(t^{\frac{19}{116}}\log^{\frac{1}{58}}t\big). \tag{3.4}$$

To complete the proof we use the inequalities

$$\sum_{n=N}^{N'} \frac{1}{n^{\frac{1}{2}+it}} = O\left(N'^{\frac{5}{82}}t^{\frac{11}{82}}\right)+O\left(N^{-\frac{17}{328}}t^{\frac{61}{328}}\right) \quad \left(N > t^{\frac{11}{36}}\right) \tag{3.5}$$

and

$$\sum_{n=1}^{N} \frac{1}{n^{\frac{1}{2}+it}} = O\left(N^{\frac{3}{14}}t^{\frac{1}{14}}\right) \quad \left(N \leqslant t^{\frac{4}{9}}\right) \tag{3.6}$$

proved in 'van der Corput's method (II)', § 5, (8) and the formula after (9). The sum (3.6) is $O\left(t^{\frac{19}{116}}\right)$ if $N = \left[t^{\frac{25}{58}}\right]$; and then (3.5), with $N' = [T]$, is

$$O\left(t^{\frac{3597}{22099}}\log t\right)+O\left(t^{\frac{3113}{164.116}}\right) = O\left(t^{\frac{19}{116}}\right)$$

since $\qquad \frac{3597}{22099} < \frac{36}{220} < \frac{19}{116}, \qquad 3113 < 19 \times 164.$

The result therefore follows from the approximate functional equation.

THE LATTICE-POINTS IN A CIRCLE

By LOO-KENG HUA (*Kunming*)

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LET $R(x)$ denote the number of lattice-points inside and on the circle $u^2+v^2 = x$. It is easily proved that, as $x \to \infty$, $R(x) \sim \pi x$, and in fact that

$$R(x) = \pi x + O(x^{\alpha}) \tag{1}$$

for some values of α less than 1. It is a question of finding the lower bound, ϑ say, of the numbers α for which (1) is true. The best result hitherto obtained is that $\vartheta \leqslant \frac{15}{46}$. This was proved by Titchmarsh* in 1933. It is the purpose of the paper to prove that $\vartheta \leqslant \frac{13}{40}$. Titchmarsh's proof depends essentially on the fact that a quadratic form he uses is positive definite. In trying to sharpen the result one arrives at the difficulty that a certain quadratic form is not positive definite. But, on examination, it is found that the variables of the quadratic form are not perfectly general. For these variables so restricted we have fortunately that the values of the form are always positive.

1. Lemmas quoted from Titchmarsh's paper

LEMMA 1. *Let $a_{\mu\nu}$ be any numbers, real or complex, such that, if $s_{m,n} = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{\mu\nu}$, then $|s_{m,n}| \leqslant G$ $(1 \leqslant m \leqslant M;\ 1 \leqslant n \leqslant N)$. Let $b_{m,n}$ denote real numbers, $0 \leqslant b_{m,n} \leqslant H$, and let each of the expressions*

$$b_{m,n}-b_{m,n+1}, \qquad b_{m,n}-b_{m+1,n}, \qquad b_{m,n}-b_{m+1,n}-b_{m,n+1}+b_{m+1,n+1}$$

be of constant sign for values of m and n in question. Then

$$\left|\sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n} b_{m,n}\right| \leqslant 5GH.$$

LEMMA 2. *Let $f(x, y)$ be a real function of x and y, and*

$$S = \sum\sum e^{2\pi i f(m,n)},$$

* *Proc. London Math. Soc.* (2), 38 (1935), 96–115. I must also refer to a paper of I. Vinogradow, *Bull. Acad. Sci. U.R.S.S.* 7 (1932), 313–36, in which the error-term $O(x^{17/53+\epsilon})$ is claimed. Unfortunately there seems to be an incurable mistake contained in the proof, namely, in § 3 F and G of his paper. For he states that, after a bulky calculation, he obtained the result $\sum_m \sum_n \sum \min\{P, (E)^{-1}\}\min\{P, (F)^{-1}) \ll p_1^4 p_2^4 p_3^4 M^{2+\epsilon} P^{-2}$. But this is apparently false as we see by considering the sum of those terms with $r_1 = s_1 = 0$ (actually the partial sum formed by these terms $\gg (M/P)^2 p_1^2 p_2^4 p_3^4 P^2$).

the sum being taken over the lattice points of a region D included in the rectangle $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$. Let

$$S' = \sum\sum e^{2\pi i\{f(m+\mu,n+\nu)-f(m,n)\}}, \qquad S'' = \sum\sum e^{2\pi i\{f(m+\mu,n-\nu)-f(m,n)\}},$$

where μ and ν are integers, and S' is taken over values of m and n such that both (m,n) and $(m+\mu, n+\nu)$ belong to D; and similarly for S''. Let ρ be a positive integer not exceeding $b-a$, and let ρ' be a positive integer not exceeding $\beta-\alpha$. Then

$$S = O\left\{\frac{(b-a)(\beta-\alpha)}{(\rho\rho')^{\frac{1}{2}}}\right\} + O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'}\sum_{\mu=1}^{\rho-1}\sum_{\nu=0}^{\rho'-1}|S'|\right\}^{\frac{1}{2}}\right] + \\ + O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'}\sum_{\mu=0}^{\rho-1}\sum_{\nu=1}^{\rho'-1}|S''|\right\}^{\frac{1}{2}}\right].$$

LEMMA 2′. *If $0 < \rho \leqslant b-a$, then*

$$S = O\left\{\frac{(b-a)(\beta-\alpha)}{\rho^{\frac{1}{2}}}\right\} + O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho}\sum_{\mu=1}^{\rho-1}|S'''|\right\}^{\frac{1}{2}}\right],$$

where $$S''' = \sum\sum e^{2\pi i\{f(m+\mu,n)-f(m,n)\}}.$$

LEMMA 3. *Let $f(x,y)$ be a real differentiable function of x and y. Let $f_x(x,y)$ be a monotonic function of x for each value of y considered, and $f_y(x,y)$ be a monotonic function of y for each value of x considered. Let $|f_x| \leqslant \frac{3}{4}$, $|f_y| \leqslant \frac{3}{4}$, for $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$, where $b-a \leqslant l$, $\beta-\alpha \leqslant l$ $(l \geqslant 1)$. Let D be the rectangle $(a,b;\alpha,\beta)$, or part of the rectangle cut off by a continuous monotonic curve. Then*

$$\sum\sum_D e^{2\pi i f(m,n)} = \iint_D e^{2\pi i f(x,y)}\,dxdy + O(l). \tag{2}$$

LEMMA 4. *Suppose that $f(x,y)$ is a real function of x and y with continuous partial derivatives of as many orders as may be required in the rectangle $(a,b;\alpha,\beta)$, and also that any curve defined by equating to zero a polynomial of given degree in these derivatives has $O(1)$ intersections with any other such curve, or with any straight line. Let $b-a \leqslant l$, $\beta-\alpha \leqslant l$. Let*

$$|f_{xx}| < AR, \qquad |f_{yy}| < AR, \qquad |f_{xy}| < AR \tag{3}$$

(A denotes a positive absolute constant, not necessarily the same one at each occurrence) and

$$|f_{xx}f_{yy} - f_{xy}^2| \geqslant r^2, \tag{4}$$

where $0 < r \leqslant R$, *throughout the rectangle. Let* $|f_x| \leqslant r_1$, $|f_y| \leqslant r_1$, $|f_{xxy}| \leqslant r_3$, $|f_{xyy}| \leqslant r_3$, $|f_{yyy}| \leqslant r_3$, *and let*

$$r_1 r_3 < K_1 r^2 \tag{5}$$

and

$$l r_3 < K_2 r, \tag{6}$$

where K_1 *and* K_2 *are sufficiently small constants. Then*

$$\int_a^b \int_\alpha^\beta e^{2\pi i f(x,y)}\, dxdy = O\left(\frac{1+|\log l|+|\log R|}{r}\right).$$

The lemmas 1, 2, 3, 4 correspond to the lemmas α, β, γ, ζ of Titchmarsh respectively.

2. Let

$$\Delta f(u,v) = f(u+m_1+m_2+m_3, v+n_1+n_2+n_3) - \\ - \sum f(u+m_1+m_2, v+n_1+n_2) + \sum f(u+m_1, v+n_1) - f(u,v),$$

and let

$$X = 6m_1 m_2 m_3, \qquad Y = 2\sum m_1 m_2 n_3,$$
$$Z = 2\sum m_1 n_2 n_3, \qquad W = 6n_1 n_2 n_3.$$

Then we have

$$\Delta u = \Delta v = 0, \qquad \Delta u^2 = \Delta uv = \Delta v^2 = 0,$$
$$\Delta u^3 = X, \qquad \Delta u^2 v = Y, \qquad \Delta uv^2 = Z, \qquad \Delta v^3 = W.$$

It is easy to prove that

$$\Delta(u^\lambda v^{k-\lambda})_{u=0,\,v=0} = O\{\eta^{k-3}(|X|+|Y|+|Z|+|W|)\}, \quad \text{for } 0 \leqslant \lambda \leqslant k,$$

if $m_i = O(\eta)$ and $n_i = O(\eta)$ for $i = 1, 2, 3$.

Let $|u_1| \leqslant \eta$, $|u_2| \leqslant \eta$, $\max(u,v) \geqslant L$. Formally, we have

$$\frac{\partial^2}{\partial u^2}\sqrt{\{(u+u_1)^2+(v+v_1)^2\}} = \frac{(v+v_1)^2}{\{(u+u_1)^2+(v+v_1)^2\}^{\frac{3}{2}}}$$
$$= \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}}\left(1+\frac{2v_1}{v}+\frac{v_1^2}{v^2}\right)\left(1+2\frac{uu_1+vv_1}{u^2+v^2}+\frac{u_1^2+v_1^2}{u^2+v^2}\right)^{-\frac{3}{2}}$$
$$= \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}}\left(1+\frac{2v_1}{v}+\frac{v_1^2}{v^2}\right)\left\{1-3\frac{uu_1+vv_1}{u^2+v^2}-\frac{3}{2}\frac{u_1^2+v_1^2}{u^2+v^2}+\right.$$
$$+\frac{15}{2}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^2+\frac{15}{2}\frac{(uu_1+vv_1)(u_1^2+v_1^2)}{(u^2+v^2)^2}-$$
$$-\frac{35}{2}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^3+\frac{15}{8}\left(\frac{u_1^2+v_1^2}{u^2+v^2}\right)^2-\frac{105}{4}\frac{(uu_1+vv_1)^2(u_1^2+v_1^2)}{(u^2+v^2)^3}+$$
$$\left.+\frac{315}{8}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^4+\dots\right\}. \tag{7}$$

Let $G(u,v) = \Delta\{\sqrt{(u^2+v^2)}\}$. Then, by (7), we have

$$G_{uu} = \frac{v^2}{(u^2+v^2)^{\frac{5}{2}}}\left\{\frac{15}{2}\,\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\right.$$

$$-\frac{35}{2}\,\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}-3\,\frac{Y+W}{v}+$$

$$\left.+15\frac{Yu^2+2Zuv+Wv^2}{v(u^2+v^2)}-3\,\frac{Zu+Wv}{v^2}\right\}+O\left(\frac{(|X|+|Y|+|Z|+|W|)\eta}{L^5}\right).$$

Similarly, we have

$$G_{vv} = \frac{u^2}{(u^2+v^2)^{\frac{5}{2}}}\left\{\frac{15}{2}\,\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\right.$$

$$-\frac{35}{2}\,\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}-3\,\frac{X+Z}{u}+$$

$$\left.+15\frac{Xu^2+2Yuv+Zv^2}{u(u^2+v^2)}-3\,\frac{Xu+Yv}{u^2}\right\}+O\left(\frac{(|X|+|Y|+|Z|+|W|)\eta}{L^5}\right)$$

and

$$G_{uv} = -\frac{uv}{(u^2+v^2)^{\frac{5}{2}}}\left\{\frac{15}{2}\,\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\right.$$

$$-\frac{35}{2}\,\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}-\frac{3}{2}\left(\frac{X+Z}{u}+\frac{Y+W}{v}\right)+$$

$$\left.+\frac{15}{2}\left(\frac{Xu^2+2Yuv+Zv^2}{u(u^2+v^2)}+\frac{Yu^2+2Zuv+Wv^2}{v(u^2+v^2)}\right)-3\,\frac{Yu+Zv}{uv}\right\}+$$

$$+O\left(\frac{(|X|+|Y|+|Z|+|W|)\eta}{L^5}\right).$$

Hence we have

$$G_{uu}\,G_{vv}-G_{uv}^2 = \frac{u^2v^2}{(u^2+v^2)^5}\left[\left\{\frac{15}{2}\,\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\right.\right.$$

$$-\frac{35}{2}\,\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}-3\,\frac{Y+W}{v}+$$

$$\left.+15\frac{Yu^2+2Zuv+Wv^2}{v(u^2+v^2)}-3\,\frac{Zu+Wv}{v^2}\right\}\times$$

$$\times\left\{\frac{15}{2}\,\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\frac{35}{2}\,\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}-\right.$$

$$\left.-3\,\frac{X+Z}{u}+15\frac{Xu^2+2Yuv+Zv^2}{u(u^2+v^2)}-3\,\frac{Xu+Yv}{u^2}\right\}-$$

$$-\left\{\frac{15}{2}\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\frac{35}{2}\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}-\right.$$
$$-\frac{3}{2}\left(\frac{X+Z}{u}+\frac{Y+W}{v}\right)+$$
$$\left.+\frac{15}{2}\left(\frac{Xu^2+2Yuv+Zv^2}{u(u^2+v^2)}+\frac{Yu^2+2Zuv+Wv^2}{v(u^2+v^2)}\right)-3\frac{Yu+Zv}{uv}\right\}^2\Bigg]+$$
$$+O\left(\frac{(X^2+Y^2+Z^2+W^2)\eta}{L^9}\right)$$
$$=\frac{u^2v^2}{(u^2+v^2)^5}\Bigg[\left(\frac{15}{2}\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\right.$$
$$\left.-\frac{35}{2}\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}\right)\times$$
$$\times\left(-3\frac{Xu+Yv}{u^2}-3\frac{Zu+Wv}{v^2}+6\frac{Yu+Zv}{uv}\right)+$$
$$+9\left(\frac{X+Z}{u}-5\frac{Xu^2+2Yuv+Zv^2}{u(u^2+v^2)}+\frac{Xu+Yv}{u^2}\right)\times$$
$$\times\left(\frac{Y+W}{v}-5\frac{Yu^2+2Zuv+Wv^2}{v(u^2+v^2)}+\frac{Zu+Wv}{v^2}\right)-$$
$$-\frac{9}{4}\left\{\frac{X+Z}{u}+\frac{Y+W}{v}-5\left(\frac{Xu^2+2Yuv+Zv^2}{u(u^2+v^2)}+\frac{Yu^2+2Zuv+Wv^2}{v(u^2+v^2)}\right)+\right.$$
$$\left.+2\frac{Yu+Zv}{uv}\right\}^2\Bigg]+O\left(\frac{(X^2+Y^2+Z^2+W^2)\eta}{L^9}\right)$$
$$=-\frac{3}{4(u^2+v^2)^7}Q(X,Y,Z,W)+O\left(\frac{(X^2+Y^2+Z^2+W^2)\eta}{L^9}\right),\quad\text{say.}$$

Then

$$-\tfrac{3}{4}Q(X,Y,Z,W)=-3[\tfrac{15}{2}(u^2+v^2)\{(X+Z)u+(Y+W)v\}-$$
$$-\tfrac{35}{2}(Xu^3+3Yu^2v+3Zuv^2+Wv^3)]\times$$
$$\times\{v^2(Xu+Yv)+u^2(Zu+Wv)-2uv(Yu+Zv)\}+$$
$$+9\{u(u^2+v^2)(X+Z)-5u(Xu^2+2Yuv+Zv^2)+(u^2+v^2)(Xu+Yv)\}\times$$
$$\times\{v(u^2+v^2)(Y+W)-5v(Yu^2+2Zuv+Wv^2)+(u^2+v^2)(Zu+Wv)\}-$$
$$-\tfrac{9}{4}\{v(u^2+v^2)(X+Z)+u(u^2+v^2)(Y+W)-$$
$$-5v(Xu^2+2Yuv+Zv^2)-5u(Yu^2+2Zuv+Wv^2)+$$
$$+2(u^2+v^2)(Yu+Zv)\}^2$$
$$=-3\{(-10u^3+\tfrac{15}{2}uv^2)X+(-45u^2v+\tfrac{15}{2}v^3)Y+$$
$$+(\tfrac{15}{2}u^3-45uv^2)Z+(\tfrac{15}{2}u^2v-10v^3)W\}\times$$

$$\times\{uv^2X-(2u^2v-v^3)Y-(-u^3+2uv^2)Z+u^2vW\}+$$
$$+9\{(-3u^3+2uv^2)X+(-9u^2v+v^3)Y+(u^3-4uv^2)Z\}\times$$
$$\times\{(-4u^2v+v^3)Y+(u^3-9uv^2)Z+(2u^2v-3v^3)W\}-$$
$$-\tfrac{9}{4}\{(-4u^2v+v^3)X+(-2u^3-7uv^2)Y+(-7u^2v-2v^3)Z+$$
$$+(u^3-4uv^2)W\}^2$$
$$= -\tfrac{3}{4}(8u^4+6u^2v^2+3v^4)v^2X^2-\tfrac{9}{4}(4u^6+4u^4v^2+21u^2v^4+6v^6)Y^2-$$
$$-\tfrac{9}{4}(6u^6+21u^4v^2+4u^2v^4+4v^6)Z^2-\tfrac{3}{4}(3u^4+6u^2v^2+8v^4)u^2W^2-$$
$$-\tfrac{3}{2}uv(-8u^4+4u^2v^2-3v^4)XY-\tfrac{3}{2}uv(-3u^4+4u^2v^2-8v^4)ZW+$$
$$+\tfrac{3}{2}(2u^6+20u^4v^2+9u^2v^4+6v^6)XZ+$$
$$+\tfrac{3}{2}(6u^6+9u^4v^2+20u^2v^4+2v^6)YW-$$
$$-\tfrac{3}{2}(4u^4+3u^2v^2+4v^4)uvXW+\tfrac{135}{2}u^3v^3YZ.$$

Therefore, we have

$$Q(X,Y,Z,W)$$
$$= (8u^4+6u^2v^2+3v^4)v^2X^2+3(4u^6+4u^4v^2+21u^2v^4+6v^6)Y^2+$$
$$+3(6u^6+21u^4v^2+4u^2v^4+4v^6)Z^2+(3u^4+6u^2v^2+8v^4)u^2W^2-$$
$$-2uv(8u^4-4u^2v^2+3v^4)XY-2uv(3u^4-4u^2v^2+8v^4)ZW-$$
$$-2(2u^6+20u^4v^2+9u^2v^4+6v^6)XZ-$$
$$-2(6u^6+9u^4v^2+20u^2v^4+2v^6)YW+$$
$$+2(4u^4+3u^2v^2+4v^4)uvXW-90u^3v^3YZ. \quad (8)$$

3. We put $n_1 = 0$. Then $W = 0$. We have then

$$Q(X,Y,Z,0)$$
$$= (8u^4+6u^2v^2+3v^4)v^2X^2+3(4u^6+4u^4v^2+21u^2v^4+6v^6)Y^2+$$
$$+3(6u^6+21u^4v^2+4u^2v^4+4v^6)Z^2-2uv(8u^4-4u^2v^2+3v^4)XY-$$
$$-2(2u^6+20u^4v^2+9u^2v^4+6v^6)XZ-90u^3v^3YZ.$$

It is the object of the section to prove that

$$Q(X,Y,Z,0) \geqslant \tfrac{1}{10}\{(u^2+v^2)^2v^2X^2+(u^2+v^2)^3(Y^2+Z^2)\}.$$

Thus, for $v \geqslant u$,

$$Q(X,Y,Z,0) \geqslant \tfrac{1}{20}(u^2+v^2)^3(X^2+Y^2+Z^2).$$

Evidently we have

$$Y^2 = 4m_1^2(m_2n_3+m_3n_2)^2 \geqslant 16m_1^2m_2m_3n_2n_3 = \tfrac{4}{3}XZ. \quad (9)$$

Therefore

$$
\begin{aligned}
&Q(X,Y,Z,0)\\
&\geqslant (8u^4+6u^2v^2+3v^4)v^2X^2+9(u^6-2u^4v^2+\tfrac{11}{2}u^2v^4+v^6)Y^2+\\
&\qquad +3(6u^6+21u^4v^2+4u^2v^4+4v^6)Z^2-\\
&\qquad\qquad -2uv(8u^4-4u^2v^2+3v^4)XY-90u^3v^3YZ\\
&\geqslant \tfrac{1}{10}\{(u^2+v^2)^2v^2X^2+(u^2+v^2)^3(Y^2+Z^2)\}+\\
&\qquad +(\tfrac{79}{10}u^4+\tfrac{58}{10}u^2v^2+\tfrac{29}{10}v^4)v^2X^2-2uv(8u^4-4u^2v^2+3v^4)XY+\\
&\qquad\qquad +(\tfrac{89}{10}u^4-\tfrac{183}{10}u^2v^2+\tfrac{312}{10}v^4)u^2Y^2+\\
&\qquad +(18u^2+\tfrac{89}{10}v^2)v^4Y^2+(\tfrac{179}{10}u^2+\tfrac{627}{10}v^2)u^4Z^2-90u^3v^3YZ\\
&\geqslant \tfrac{1}{10}\{(u^2+v^2)^2v^2X^2+(u^2+v^2)^3(Y^2+Z^2)\},
\end{aligned}
$$

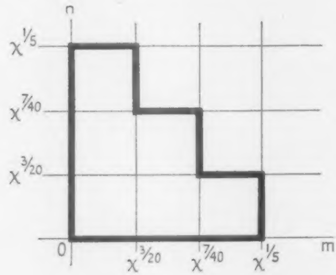
since

$$
\begin{aligned}
&(79u^4+58u^2v^2+29v^4)(89u^4-183u^2v^2+312v^4)-10^2(8u^4-4u^2v^2+3v^4)^2\\
&\qquad\qquad \geqslant 600u^8-3000u^6v^2+10000u^4v^4 \geqslant 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(180u^2+89v^2)(179u^2+627v^2)-450^2u^2v^2\\
&\qquad \geqslant \{2\surd(180.179.89.627)+179.89+180.627-450^2)u^2v^2 \geqslant 0.
\end{aligned}
$$

4. It is known that



$$
\int_0^x \{R(y)-\pi y\}\,dy = \frac{x}{\pi}\sum_{\nu=1}^{\infty}\frac{r(\nu)}{\nu}J_2\{2\pi\surd(\nu x)\},
$$

where $r(\nu)$ is the number of solutions of the Diophantine equation

$$
x^2+y^2=\nu.
$$

Evidently, we have

$$
\int_0^x \{R(y)-\pi y\}\,dy = \frac{4x}{\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{J_2\{2\pi\surd[(m^2+n^2)x]\}}{m^2+n^2}.
$$

Let C denote the region bounded by the heavy lines in the figure, and C' denote the remaining part in the first quadrant. It is easy

to deduce that, if $0 < \alpha < 1$ (we shall put $\alpha = \frac{13}{40}$ later),

$$\int_x^{x\pm x^\alpha} \{R(y)-\pi y\}\, dy = 4\int_x^{x\pm x^\alpha} \sum_C\sum \frac{\sqrt{y}J_1\{2\pi\sqrt{[(m^2+n^2)y]}\}}{(m^2+n^2)^{\frac{1}{2}}}\, dy +$$

$$+\frac{4}{\pi}\left\{\sum_{C'}\sum \frac{yJ_2\{2\pi\sqrt{[(m^2+n^2)y]}\}}{m^2+n^2}\right\}_x^{x\pm x^\alpha}$$

$$= \sum\nolimits_1 + \{\sum\nolimits_2\}_x^{x\pm x^\alpha}, \quad \text{say.}$$

Now $$J_1\{2\pi\sqrt{(\nu y)}\} = \frac{\sin\{2\pi\sqrt{(\nu y)}-\frac{1}{4}\pi\}}{\pi(\nu y)^{\frac{1}{4}}} + O\left(\frac{1}{(\nu y)^{\frac{3}{4}}}\right).$$

Hence $$\sum\nolimits_1 = O\left(\int_x^{x\pm x^\alpha} |\phi(y)|y^{\frac{1}{4}}\, dy\right) + O(x^{\alpha-\frac{1}{4}}),$$

where

$$\phi(y) = \sum_C\sum \frac{e^{2\pi i\sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{3}{4}}} \quad (x-x^\alpha \leqslant y \leqslant x+x^\alpha). \tag{10}$$

Similarly, we have

$$\sum\nolimits_2 = O\{y^{\frac{3}{4}}|\psi(y)|\} + O(x^{\frac{1}{4}}),$$

where

$$\psi(y) = \sum_{C'}\sum \frac{e^{2\pi i\sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{5}{4}}} \quad (x-x^\alpha \leqslant y \leqslant x+x^\alpha). \tag{11}$$

If $m \leqslant x^{\frac{3}{20}}$, then

$$y^{\frac{1}{4}}\left|\sum_{\substack{C'\\ m\leqslant x^{\frac{3}{20}}}}\sum \frac{e^{2\pi i\sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{3}{4}}}\right| \leqslant y^{\frac{1}{4}} \sum_{m\leqslant x^{\frac{3}{20}}} \sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^{\frac{3}{4}}}$$

$$\leqslant y^{\frac{1}{4}} \sum_{m\leqslant x^{\frac{3}{20}}} \left(\sum_{n=1}^{m} \frac{1}{n^{\frac{3}{2}}} + \sum_{n=m+1}^{\infty} \frac{1}{n^{\frac{3}{2}}}\right)$$

$$= O\left(y^{\frac{1}{4}} \sum_{m\leqslant x^{\frac{3}{20}}} m^{-\frac{1}{2}}\right) = O\left(x^{\frac{1}{4}+\frac{3}{40}}\right) = O(x^\alpha).$$

The same result holds for $n \leqslant x^{\frac{3}{20}}$.

If $m \geqslant x^{\frac{1}{5}}$, then

$$y^{\frac{3}{4}}\left|\sum_{\substack{C'\\ m>x^{\frac{1}{5}}}}\sum \frac{e^{2\pi i\sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{5}{4}}}\right| \leqslant y^{\frac{3}{4}} \sum_{m>x^{\frac{1}{5}}} O\left(\frac{1}{m^{\frac{3}{2}}}\right) = O\left(x^{\frac{3}{4}-\frac{1}{10}}\right) = O(x^{2\alpha}).$$

The same result holds for $n \geqslant x^{\frac{1}{5}}$.

Let D be the region common to C and the square

$$x^{\frac{3}{20}} \leqslant m, n \leqslant x^{\frac{1}{5}}$$

and D' be the remaining part of the square. Thus

$$y^{\frac{1}{4}}|\phi(y)| = y^{\frac{1}{4}}\left|\sum_D\sum \frac{e^{2\pi i\sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{3}{4}}}\right| + O(x^\alpha) \tag{12}$$

and

$$y^{\frac{3}{4}}|\psi(y)| = y^{\frac{3}{4}}\left|\sum_{D'}\sum \frac{e^{2\pi i\sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{5}{4}}}\right| + O(x^{2\alpha}). \tag{13}$$

5. Now we consider a sum of the form

$$S = \sum_{m=M}^{M'}\sum_{n=N}^{N'} e^{2\pi i f(m,n)}, \qquad M \leqslant M' \leqslant 2M;\ N \leqslant N' \leqslant 2N$$

where $f(m,n) = \sqrt{\{(m^2+n^2)y\}}$. Let R denote the domain of summation and let $L = \max(m,n)$. Now the terms considered satisfy

$$x^{\frac{3}{20}} < L < x^{\frac{1}{5}}. \tag{14}$$

If $M'-M < x^{\frac{1}{8}}$, we have $S = O\big(Lx^{\frac{1}{8}}\big)$, so that, by Lemma 1,

$$x^{\frac{1}{4}}\sum_R\sum \frac{e^{2\pi i f(m,n)}}{(m^2+n^2)^{\frac{3}{4}}} = O\big(x^{\frac{1}{4}+\frac{1}{8}}L^{-\frac{1}{2}}\big) = O\big(x^{\frac{1}{4}+\frac{1}{8}-\frac{3}{40}}\big) = O(x^\alpha) \tag{15}$$

and

$$x^{\frac{3}{4}}\sum_R\sum \frac{e^{2\pi i f(m,n)}}{(m^2+n^2)^{\frac{5}{4}}} = O\big(x^{\frac{3}{4}+\frac{1}{8}}L^{-\frac{3}{2}}\big) = O\big(x^{\frac{3}{4}+\frac{1}{8}-\frac{9}{40}}\big) = O(x^{2\alpha}). \tag{16}$$

A similar result holds if $N'-N < x^{\frac{1}{8}}$. We may therefore suppose that

$$M'-M > x^{\frac{1}{8}}, \qquad N'-N > x^{\frac{1}{8}}.$$

Applying Lemma 2′ once and Lemma 2 twice, we have

$$\begin{aligned} S &= O\big(L^2\rho^{-\frac{1}{2}}\big) + O\Big[L\rho^{-\frac{1}{2}}\Big(\sum_{m_1=1}^{\rho-1}|S_1|\Big)^{\frac{1}{2}}\Big] \\ &= O\big(L^2\rho^{-\frac{1}{2}}\big) + O\Big[L^{\frac{3}{2}}\rho^{-1}\sum_{i=1}^{2}\Big\{\sum_{m_1=1}^{\rho-1}\Big(\sum_{m_2=1}^{\rho-1}\sum_{m_2=0}^{\rho-1}|S_2^{(i)}|\Big)^{\frac{1}{2}}\Big\}^{\frac{1}{2}}\Big] \\ &= O\big(L^2\rho^{-\frac{1}{2}}\big) + O\Big[L^{\frac{7}{4}}\rho^{-\frac{3}{2}}\sum_{i=1}^{4}\Big\{\sum_{m_1=1}^{\rho-1}\Big[\sum_{m_2=1}^{\rho-1}\sum_{n_2=0}^{\rho-1}\Big(\sum_{m_3=1}^{\rho^2-1}\sum_{n_3=0}^{\rho^2-1}|S_3^{(i)}|\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}}\Big], \end{aligned} \tag{17}$$

provided that

$$1 \leqslant \rho^2 \leqslant \tfrac{1}{2}x^{\frac{1}{8}}, \tag{18}$$

where

$$S_3^{(1)} = S_3 = \sum\sum e^{2\pi i g(m,n)}, \qquad g(m,n) = \sqrt{x}\,G(m,n)$$

with $W = 0$, and $S_3^{(2)}$, $S_3^{(3)}$, $S_3^{(4)}$ are corresponding sums with $(-m_3, n_3)$,

$(m_3, -n_3)$, $(-m_3, -n_3)$ for (m_3, n_3) respectively. We may suppose that $v > u$ (by symmetry). By § 3, we have

$$g_{uu}g_{vv}-g_{uv}^2 \geqslant \frac{x}{L^8}(X^2+Y^2+Z^2)+O\left(\frac{(X^2+Y^2+Z^2)x\eta}{L^9}\right)$$

$$\geqslant A\frac{x}{L^8}(X^2+Y^2+Z^2),$$

where A is a certain constant, provided that

$$\rho = o(L^{\frac{1}{2}}). \tag{19}$$

In fact, since $\eta = O(\rho^2)$,

$$\frac{(X^2+Y^2+Z^2)x\eta}{L^9} = O\left(\frac{x(X^2+Y^2+Z^2)\rho^2}{L^9}\right) = o\left(\frac{x(X^2+Y^2+Z^2)}{L^8}\right).$$

6. Since $X = O(\rho^4)$, $Y = O(\rho^4)$, $Z = O(\rho^4)$, we have

$$g_{uu} = O\left(\frac{x^{\frac{1}{2}}\rho^4}{L^4}\right)+O\left(\frac{x^{\frac{1}{2}}\rho^4\eta}{L^5}\right) = O\left(\frac{x^{\frac{1}{2}}\rho^4}{L^4}\right),$$

since $\eta = O(\rho^2) = O(L)$. A similar result holds for g_{vv} and g_{uv}. Hence, if

$$l = aL^4x^{-\frac{1}{2}}\rho^{-4}$$

with a sufficiently small a, the variation of g_u and g_v in a square of side l is less than $\frac{1}{2}$. Suppose the region of summation S_3 divided into such squares or parts of such squares. Then to each square correspond integers μ, ν such that, if

$$h(u,v) = g(u,v)-\mu u-\nu v,$$

then $|h_u| \leqslant \frac{3}{4}$, $|h_v| \leqslant \frac{3}{4}$. Hence, by Lemma 3, for each square

$$\sum\sum e^{2\pi i g(m,n)} = \iint e^{2\pi i h(u,v)}\,dudv + O(l).$$

By § 5, we can take (in Lemma 4),

$$r^2 = A\frac{x}{L^8}(X^2+Y^2+Z^2).$$

Also $$r_1 = O(1), \qquad r_3 = O(x^{\frac{1}{2}}\rho^4L^{-5}),$$

so that the condition (5) of Lemma 4 is satisfied if

$$L^3\rho^4 < K_1Ax^{\frac{1}{2}}(X^2+Y^2+Z^2). \tag{20}$$

Lemma 4 also requires that $lr_3 < K_2 r$,

i.e. $$L^6 < K_2(X^2+Y^2+Z^2)x. \tag{21}$$

Since $X^2+Y^2+Z^2 = O(\rho^8)$, (21) is satisfied if (20) is satisfied. Hence, if (20) holds, Lemma 4 gives

$$\iint e^{2\pi i h(u,v)}\,dudv = O\Big(\frac{L^4 \log x}{x^{\frac{1}{2}}(X^2+Y^2+Z^2)^{\frac{1}{2}}}\Big),$$

assuming that $L = O(x^{A})$.

The number of such terms is $O(L^2/l^2)$, provided that $l \leqslant L$. Hence

$$S_3 = O\Big(\frac{L^6 \log x}{l^2 x^{\frac{1}{2}}(X^2+Y^2+Z^2)^{\frac{1}{2}}}\Big)+O\Big(\frac{L^2}{l}\Big)$$

$$= O\Big(\frac{x^{\frac{1}{2}}\rho^8 \log x}{L^2(X^2+Y^2+Z^2)^{\frac{1}{2}}}\Big) = O\Big(\frac{x^{\frac{1}{2}}\rho^8 \log x}{L^2 m_1 m_2 m_3}\Big), \tag{22}$$

with the restriction (20). The result, of course, holds for $S_3^{(i)}$ $(i = 2,3,4)$.

7. Substituting (22) in (17), we have, for $v \geqslant u$,

$$O\Big(\frac{L^{\frac{7}{4}}}{\rho^{\frac{3}{2}}}\Big\{\sum_{m_1=1}^{\rho-1}\Big[\sum_{m_2=1}^{\rho-1}\sum_{n_2=0}^{\rho-1}\Big(\sum_{m_3=1}^{\rho^2-1}\sum_{n_3=0}^{\rho^2-1}|S_3|\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}}\Big)$$

$$= O\Big(\frac{L^{\frac{7}{4}}}{\rho^{\frac{3}{2}}}\Big(\frac{x^{\frac{1}{2}}\rho^8 \log x}{L^2}\Big)^{\frac{1}{8}}\Big\{\sum_{m_1=1}^{\rho-1}\Big[\sum_{m_2=1}^{\rho-1}\sum_{n_2=0}^{\rho-1}\Big(\sum_{m_3=1}^{\rho^2-1}\sum_{m_3=0}^{\rho^2-1}\frac{1}{m_1 m_2 m_3}\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}}\Big)$$

$$= O\big\{L^{\frac{3}{2}}\rho^{-\frac{1}{2}}x^{\frac{1}{16}}(\log x)^{\frac{1}{8}}.\rho(\log \rho)^{\frac{1}{8}}\big\}$$

$$= O\big\{L^{\frac{3}{2}}\rho^{\frac{1}{2}}x^{\frac{1}{16}}(\log x)^{\frac{1}{4}}\big\},$$

provided that (20) holds.

Next we consider the sum of those terms which do not satisfy (20); we have the inequality

$$X^2+Y^2+Z^2 = O\big(L^3\rho^4 x^{-\frac{1}{2}}\big).$$

Consequently we have $m_1 m_2 m_3 = O\big(L^3\rho^4 x^{-\frac{1}{2}}\big)$, $m_1 n_2 n_3 = O\big(L^3\rho^4 x^{-\frac{1}{2}}\big)$. Since $S_3 = O(L^2)$, the terms for which (20) is not satisfied contribute

$$O\Big(L^{\frac{7}{4}}\rho^{-\frac{3}{2}}\Big\{\sum_{m_1=1}^{\rho-1}\Big[\sum_{m_2=1}^{\rho-1}\sum_{n_2=1}^{\rho-1}\Big(\sum_{m_3=O(L^{\frac{3}{2}}\rho^2 x^{-\frac{1}{4}}m_1^{-1}m_2^{-1})}\ \sum_{n_3=O(L^{\frac{3}{2}}\rho^2 x^{-\frac{1}{4}}m_1^{-1}n_2^{-1})} 1\Big)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}}\Big)$$

$$= O\Big(L^{\frac{7}{4}}\rho^{-\frac{3}{2}}\Big\{\sum_{m_1=1}^{\rho-1}\Big[\sum_{m_2=1}^{\rho-1}\sum_{n_2=1}^{\rho-1} L^{\frac{3}{2}}\rho^2 x^{-\frac{1}{4}} m_1^{-1} m_2^{-\frac{1}{2}} n_2^{-\frac{1}{2}}\Big]^{\frac{1}{2}}\Big\}^{\frac{1}{2}}\Big)$$

$$= O\Big(L^{\frac{17}{8}}\rho^{-1}x^{-\frac{1}{16}}\Big[\sum_{m_1=1}^{\rho-1}\Big(\sum_{m_2=1}^{\rho-1}\sum_{n_2=1}^{\rho-1}\frac{1}{(m_2 n_2)^{\frac{1}{2}}}\Big)^{\frac{1}{2}}\frac{1}{m_1^{\frac{1}{2}}}\Big]^{\frac{1}{2}}\Big)$$

$$= O\big(L^{\frac{17}{8}}\rho^{-1}x^{-\frac{1}{16}}\rho^{\frac{1}{2}}\big) = O\big(L^{\frac{17}{8}}\rho^{-\frac{1}{2}}x^{-\frac{1}{16}}\big).$$

(The terms with $n_2 = 0$ contribute an insignificant order.) A similar result holds for $v < u$. Therefore

$$S = O\big(L^2\rho^{-\frac{1}{2}}\big)+o\big\{L^{\frac{3}{2}}\rho^{\frac{1}{2}}x^{\frac{1}{16}}(\log x)^{\frac{1}{4}}\big\}+O\big(L^{\frac{17}{8}}\rho^{-\frac{1}{2}}x^{-\frac{1}{16}}\big).$$

The first two terms are of the same form if

$$\rho = \left[L^{\frac{1}{2}} x^{-\frac{1}{16}} (\log x)^{-\frac{1}{4}}\right],$$

and, if this is a permissible value of ρ, we have

$$S = O\left\{L^{\frac{7}{4}} x^{\frac{1}{32}} (\log x)^{\frac{1}{8}}\right\} + O\left\{L^{\frac{15}{8}} x^{-\frac{1}{32}} (\log x)^{\frac{1}{8}}\right\} = O\left\{L^{\frac{7}{4}} x^{\frac{1}{32}} (\log x)\right\}^{\frac{1}{8}}. \quad (23)$$

Now we shall verify all the conditions. The condition (18)

$$1 \leqslant \rho^2 = L x^{-\frac{1}{8}} (\log x)^{-\frac{1}{2}} \leqslant \tfrac{1}{2} x^{\frac{1}{8}}$$

can be written $\qquad x^{\frac{1}{8}} (\log x)^{\frac{1}{2}} \leqslant L \leqslant x^{\frac{1}{4}} (\log x)^{\frac{1}{2}}.$

This is always satisfied, since $\frac{1}{8} < \frac{3}{20} < \frac{1}{5} < \frac{1}{4}$. The condition (19) is

$$L^{\frac{1}{2}} x^{-\frac{1}{16}} (\log x)^{-\frac{1}{4}} - 1 \leqslant \rho = o\left(L^{\frac{1}{2}}\right).$$

This is always satisfied.

8. By Lemma 1 and (23), we have

$$\sum_R\sum \frac{e^{2\pi i \sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{3}{4}}} = O\left\{L^{\frac{7}{4}-\frac{3}{2}} x^{\frac{1}{32}} (\log x)^{\frac{1}{8}}\right\} \quad (24)$$

and

$$\sum_R\sum \frac{e^{2\pi i \sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{5}{4}}} = O\left\{L^{\frac{7}{4}-\frac{5}{2}} x^{\frac{1}{32}} (\log x)^{\frac{1}{8}}\right\}. \quad (25)$$

Now we divide the sum

$$\sum_D\sum \frac{e^{2\pi i \sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{3}{4}}} = \sum_{p=1}^{P} \sum_{q=1}^{Q} \left\{\sum_{2^{p-1}}^{2^p} \sum_{2^{q-1}}^{2^q} \frac{e^{2\pi i \sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{3}{4}}}\right\}.$$

By (12) and (24) we have, taking $L_0 = x^{\frac{7}{40}}$,

$$\begin{aligned} y^{\frac{1}{4}} |\phi(y)| &= O\left(x^{\frac{1}{4}} \sum_{p=1}^{P} \sum_{q=1}^{Q} \{\max(2^p, 2^q)\}^{\frac{1}{4}} x^{\frac{1}{32}} (\log x)^{\frac{1}{8}}\right) + O(x^{\alpha}) \\ &= O\left\{x^{\frac{9}{32}} L_0^{\frac{1}{4}} (\log x)^{\frac{9}{8}}\right\} + O(x^{\alpha}) \\ &= O\{x^{\alpha} (\log x)^{\frac{9}{8}}\}. \end{aligned}$$

Similarly, by (13) and (25), we have

$$\begin{aligned} y^{\frac{3}{4}} |\psi(y)| &= O\left(x^{\frac{3}{4}} \sum_{p=1}^{P} \sum_{q=1}^{Q} \{\max(2^p, 2^q)\}^{-\frac{3}{4}} x^{\frac{1}{32}} (\log x)^{\frac{1}{8}}\right) + O\left(x^{\frac{1}{4}+\alpha}\right) \\ &= O\left\{x^{\frac{26}{40}} (\log x)^{\frac{9}{8}}\right\} + O\left(x^{\frac{1}{4}+\alpha}\right) \\ &= O\left\{x^{2\alpha} (\log x)^{\frac{9}{8}}\right\}. \end{aligned}$$

Thus we have

$$\int_x^{x \pm x^{\alpha}} \{R(y) - \pi y\}\, dy = O\left\{x^{2\alpha} (\log x)^{\frac{9}{8}}\right\}.$$

Hence, in the usual way, we deduce easily

$$R(x) = \pi x + O\left\{x^{\frac{13}{40}} (\log x)^{\frac{9}{8}}\right\}.$$

SUMMATION FORMULAE AND SELF-RECIPROCAL FUNCTIONS (III)

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1. Introduction

In several previous papers I have discussed the connexion between certain self-reciprocal functions and summation formulae.*

The results were also connected with Dirichlet's series such as that for the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which satisfies the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s)\zeta(1-s).$$

Now the Dirichlet's series

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n)n^{-s} \tag{1.1}$$

is of importance in the theory of prime numbers.† Further, it satisfies the functional equation

$$-\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(1-s) = \frac{\Gamma'}{\Gamma}(s) - \tfrac{1}{2}\pi\tan\tfrac{1}{2}s\pi - \log 2\pi. \tag{1.2}$$

This suggests that results analogous to those of the previous papers may also be derived from this Dirichlet's series.

In the present paper I discuss several such results. The results obtained are all equivalent to previously known formulae in the theory of prime numbers, but I now show how these formulae may be regarded as results in the theory of transforms.

* A. P. Guinand, *Proc. London Math. Soc.* (2), 43 (1937), 439–48, referred to as (A); *Quart. J. of Math.* (Oxford), 9 (1938), 53–67, referred to as (I); ibid. 10 (1939), 38–44, referred to as (B); ibid. 10 (1939), 104–18, referred to as (II); and *J. of London Math. Soc.* 14 (1939), 97–100, referred to as (C).

† A. E. Ingham, *The Distribution of Prime Numbers* (Cambridge, 1932), 17 and 73, where $\Lambda(n) = \log p$ if n is a power of a prime p, otherwise $\Lambda(n) = 0$.

2. Preliminary results

By analogy with the results of (A) and (I) we expect to find, corresponding to (1.2), a self-reciprocal function with respect to the transformation of Fourier's type for which*

$$\mathfrak{K}(s) = \frac{s}{s-1}.$$

This gives rise to the transformation

$$T_0\{f(x)\} = \frac{1}{x}f\left(\frac{1}{x}\right) - \frac{1}{x}\int\limits_{1/x}^{\infty} f(t)\,\frac{dt}{t}, \tag{2.1}$$

which was discussed in (B). Another transformation used in the sequel is obtained if we put

$$\mathfrak{K}(s) = \frac{s-\alpha}{s+\alpha-1}.$$

We arrive at the transformation

$$T_\alpha\{f(x)\} = \frac{1}{x}f\left(\frac{1}{x}\right) + (2\alpha-1)x^{\alpha-1}\int\limits_{1/x}^{\infty} t^{\alpha-1}f(t)\,dt \tag{2.2}$$

if $R(\alpha) < \frac{1}{2}$, and

$$T_\alpha\{f(x)\} = \frac{1}{x}f\left(\frac{1}{x}\right) + (1-2\alpha)x^{\alpha-1}\int\limits_{0}^{1/x} t^{\alpha-1}f(t)\,dt \tag{2.3}$$

if $R(\alpha) > \frac{1}{2}$.

We also need the following results:†

$$\sum_{n\leqslant x}{}' \Lambda(n) = x - \log 2\pi - \sum_{\rho} \frac{x^\rho}{\rho} - \tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right) \quad (x > 1), \tag{2.4}$$

$$\sum_{n\leqslant x}{}' \frac{\Lambda(n)}{n} = \log x - \gamma + \sum_{\rho} \frac{x^{-\rho}}{\rho} - \frac{1}{x} + \tfrac{1}{2}\log\left(\frac{x+1}{x-1}\right) \quad (x > 1). \tag{2.5}$$

The sums on the right-hand sides are taken over the zeros ρ of $\zeta(s)$ in the critical strip, and are to be interpreted as

$$\lim_{T\to\infty} \sum_{-T<I(\rho)<T} \frac{x^{\pm\rho}}{\rho}.$$

* We are using the notation of E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), Chapter VIII.

† A. E. Ingham, loc. cit. 77 and 81. The formula (2.5) appears to have been stated incorrectly in this paper.

Further,
$$\sum_{n\leqslant x}{}' \Lambda(n) = x+O(xe^{-a\sqrt{(\log x)}}), \tag{2.6}$$

$$\sum_{n\leqslant x}{}' \frac{\Lambda(n)}{n} = \log x-\gamma+O(e^{-a\sqrt{(\log x)}}), \tag{2.7}$$

where a is some positive constant.* An immediate consequence of (2.7) is
$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{\Lambda(n)}{n}-\log N\right\} = -\gamma. \tag{2.8}$$

3. The self-reciprocal function

By analogy with (A) and (I) we expect to find that a function involving
$$\phi_1(x) = \frac{1}{x}\Big\{\sum_{n\leqslant x}{}' \Lambda(n)-x\Big\} \tag{3.1}$$

is self-reciprocal with respect to T_0.

Put
$$F(x) = \phi_1(x)+\frac{1}{2}\sum_{\rho}\frac{x^{\rho-1}}{\rho}+A(x) \tag{3.2}$$

where
$$A(x) = \begin{cases} \dfrac{1}{4x}\log\dfrac{1+x}{1-x} & (x<1), \\[2ex] \dfrac{1}{2x}(\log 2\pi+\gamma)+\dfrac{1}{4x}\log(x^2-1) & (x>1). \end{cases} \tag{3.3}$$

We can readily verify that
$$T_0\{A(x)\} = -A(x). \tag{3.4}$$

Further, the transform of $\phi_1(x)$ is
$$T_0\{\phi_1(x)\} = \frac{1}{x}\phi_1\Big(\frac{1}{x}\Big)-\frac{1}{x}\int_{1/x}^{\infty}\phi_1(t)\frac{dt}{t}.$$

It follows from (2.6) that the integral is absolutely convergent since it is
$$O\Big\{\int^{\infty} e^{-a\sqrt{(\log t)}}\frac{dt}{t}\Big\} = O\Big\{\int^{\infty} ze^{-az}\,dz\Big\}.$$

Hence, if N is an integer and $y = 1/x$,
$$T_0\{\phi_1(x)\} = \Big\{\sum_{n\leqslant 1/x}{}' \Lambda(n)-\frac{1}{x}\Big\}-\frac{1}{x}\int_{1/x}^{\infty}\Big\{\sum_{n\leqslant t}{}'\Lambda(n)-t\Big\}\frac{dt}{t^2}$$

* A. E. Ingham, loc. cit. 65 for (2.6). We may prove (2.7) in the same way.

$$=\sum_{n\leqslant y}{}' \Lambda(n)-y-y\int\limits_{y}^{[y+1]}\Big\{\sum_{n\leqslant t}{}'\Lambda(n)-t\Big\}\frac{dt}{t^2}-$$

$$-y\int\limits_{[y+1]}^{N+1}\Big\{\sum_{n\leqslant t}{}'\Lambda(n)-t\Big\}\frac{dt}{t^2}-y\int\limits_{N+1}^{\infty}\phi_1(t)\frac{dt}{t}$$

$$=\sum_{n\leqslant y}{}' \Lambda(n)-y-y\Big\{\sum_{n=1}^{[y]}\Lambda(n)\Big\}\int\limits_{y}^{[y+1]}\frac{dt}{t^2}-$$

$$-y\sum_{n=[y+1]}^{N}\{\Lambda(1)+\Lambda(2)+\ldots+\Lambda(n)\}\int\limits_{n}^{n+1}\frac{dt}{t^2}+$$

$$+y\int\limits_{y}^{N+1}\frac{dt}{t}-y\int\limits_{N+1}^{\infty}\phi_1(t)\frac{dt}{t}$$

$$=\sum_{n\leqslant y}{}' \Lambda(n)-y-\Big\{\sum_{n=1}^{[y]}\Lambda(n)\Big\}\Big\{1-\frac{y}{[y+1]}\Big\}-$$

$$-y\sum_{n=[y+1]}^{N}\{\Lambda(1)+\Lambda(2)+\ldots+\Lambda(n)\}\Big\{\frac{1}{n}-\frac{1}{n+1}\Big\}+$$

$$+y\log\frac{N+1}{y}-y\int\limits_{N+1}^{\infty}\phi_1(t)\frac{dt}{t}$$

$$=\sum_{n\leqslant y}{}' \Lambda(n)-y-\Big\{\sum_{n=1}^{[y]}\Lambda(n)\Big\}\Big\{1-\frac{y}{[y+1]}\Big\}-$$

$$-\frac{y}{[y+1]}\sum_{n=1}^{[y+1]}\Lambda(n)+\frac{y}{N+1}\sum_{n=1}^{N}\Lambda(n)-$$

$$-y\sum_{n=[y+2]}^{N}\frac{\Lambda(n)}{n}+y\log\frac{N+1}{y}-y\int\limits_{N+1}^{\infty}\phi_1(t)\frac{dt}{t}$$

$$=\sum_{n\leqslant y}{}' \Lambda(n)-y-\sum_{n=1}^{[y]}\Lambda(n)-\frac{y}{[y+1]}\Lambda([y+1])+$$

$$+\frac{y}{N+1}\sum_{n=1}^{N}\Lambda(n)-y\Big\{\sum_{n=1}^{N}\frac{\Lambda(n)}{n}-\log(N+1)\Big\}+$$

$$+y\sum_{n=1}^{[y+1]}\frac{\Lambda(n)}{n}-y\int\limits_{N+1}^{\infty}\phi_1(t)\frac{dt}{t}.$$

If we make N tend to infinity this becomes

$$-\tfrac{1}{2}\Lambda(y)+y\left\{\sum_{n=1}^{[y+1]}\frac{\Lambda(n)}{n}-\log y\right\}-\frac{y}{[y+1]}\Lambda([y+1])+\gamma y,$$

where $\Lambda(y)$ is zero if y is not an integer. This is equal to

$$y\left\{\sum_{n\leqslant y}{}' \Lambda(n)-\log y+\gamma\right\}=\frac{1}{x}\left\{\sum_{n\leqslant 1/x}{}' \Lambda(n)-\log\frac{1}{x}+\gamma\right\}$$
$$=\phi_2(x) \quad \text{say.}$$

Similarly, we can show that

$$T_0\{\phi_2(x)\}=\phi_1(x).$$

Now, by (2.5),

$$\phi_2(x)=\begin{cases}\displaystyle\sum_\rho\frac{x^{\rho-1}}{\rho}-1+\frac{1}{2x}\log\frac{1+x}{1-x} & (x<1),\\[2ex] \displaystyle\frac{\gamma}{x}+\frac{1}{x}\log x & (x>1),\end{cases}$$

and by (2.4) we can write this as

$$\phi_2(x)=\begin{cases}\displaystyle\sum_\rho\frac{x^{\rho-1}}{\rho}-1+\frac{1}{2x}\log\frac{1+x}{1-x} \quad (x<1),\\[2ex] \displaystyle\frac{1}{x}\left\{\sum_{n\leqslant x}{}' \Lambda(n)-x\right\}+\sum_\rho\frac{x^{\rho-1}}{\rho}+\frac{1}{x}(\log 2\pi+\gamma)+\frac{1}{2x}\log(x^2-1) \\ \hfill (x>1)\end{cases}$$

$$=\frac{1}{x}\left\{\sum_{n\leqslant x}{}' \Lambda(n)-x\right\}+\sum_\rho\frac{x^{\rho-1}}{\rho}+\begin{cases}\displaystyle\frac{1}{2x}\log\frac{1+x}{1-x} \quad (x<1),\\[2ex] \displaystyle\frac{1}{x}(\log 2\pi+\gamma)+\frac{1}{2x}\log(x^2-1) \\ \hfill (x>1)\end{cases}$$

$$=\phi_1(x)+\sum_\rho\frac{x^{\rho-1}}{\rho}+2A(x).$$

Hence the transform of

$$F(x)-\frac{1}{2}\sum_\rho\frac{x^{\rho-1}}{\rho}=\phi_1(x)+A(x)$$

is

$$\phi_2(x)-A(x)=\phi_1(x)+\sum_\rho\frac{x^{\rho-1}}{\rho}+A(x)$$
$$=F(x)+\frac{1}{2}\sum_\rho\frac{x^{\rho-1}}{\rho}.$$

Also, the transform of

$$F(x)+\frac{1}{2}\sum_{\rho}\frac{x^{\rho-1}}{\rho} = \phi_2(x)-A(x)$$

is

$$\phi_1(x)+A(x) = F(x)-\frac{1}{2}\sum_{\rho}\frac{x^{\rho-1}}{\rho}.$$

Adding these results, it follows that $F(x)$ is self-reciprocal with respect to T_0, and we have

THEOREM 1. *The function $F(x)$ defined by* (3.2) *and* (3.3) *is self-reciprocal with respect to the transformation T_0: that is*

$$F(x) = \frac{1}{x}F\left(\frac{1}{x}\right)-\frac{1}{x}\int\limits_{1/x}^{\infty} F(t)\,\frac{dt}{t}.$$

We would expect to be able to derive a summation formula from this result by the methods of (I). This cannot readily be done without a series of involved assumptions, so I shall only derive a formal result which will indicate the type of formula to be expected. Any particular case of the result can be investigated separately, and I give an example to illustrate this.

Suppose that $f(x)$ tends to zero as x tends to infinity, that

$$g(x) = -\frac{1}{x}f\left(\frac{1}{x}\right),$$

and that $f(x)$ and $g(x)$ are the integrals of their derivatives. Then

$$\begin{aligned} xg'(x) &= \frac{1}{x^2}f'\left(\frac{1}{x}\right)+\frac{1}{x}f\left(\frac{1}{x}\right) \\ &= \frac{1}{x}\left\{\frac{1}{x}f'\left(\frac{1}{x}\right)\right\}-\frac{1}{x}\int\limits_{1/x}^{\infty} tf'(t)\,\frac{dt}{t} \\ &= T_0\{xf'(x)\}. \end{aligned}$$

By the Parseval theorem for T_0 transforms we have, formally,

$$\int\limits_0^{\infty} xf'(x)F(x)\,dx = \int\limits_0^{\infty} xg'(x)F(x)\,dx.$$

From this result we can derive a formal summation formula by a

series of partial integrations, as in (I). The final result may be written in the form

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\Lambda(n)f(n)-\int_0^N f(x)\,dx\right\}=\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\Lambda(n)g(n)-\int_0^N g(x)\,dx\right\}-$$

$$-\tfrac{1}{2}(\log 2\pi+\gamma)f(1)-\frac{1}{2}\int_0^\infty\frac{f(x)}{1+x}\,dx-$$

$$-\frac{1}{2}\int_0^2\frac{f(x)-f(1)}{|x-1|}\,dx-\frac{1}{2}\int_2^\infty\frac{f(x)}{x-1}\,dx-$$

$$-\sum_\rho\int_0^\infty f(x)x^{\rho-1}\,dx. \tag{3.5}$$

For example, if we put $f(x)=e^{-ax}$ in (3.5), we get

$$\sum_{n=1}^\infty\Lambda(n)e^{-an}=\frac{1}{a}+\gamma\sinh a-\log a+e^{-a}\log\frac{a}{2\pi}+$$

$$+\int_0^a\cosh(t-a)\log t\,dt+\sum_{n=1}^\infty\frac{\Lambda(n)}{n}(1-e^{-a/n})-\sum_\rho\Gamma(\rho)a^{-\rho}.$$

This result may be verified by considering the integral

$$\frac{1}{2\pi i}\int\left\{-\frac{\zeta'}{\zeta}(s)\right\}\Gamma(s)a^{-s}\,ds$$

around the contour $2\pm iT$, $-\frac{1}{2}\pm iT$.

4. Further self-reciprocal functions and pairs of transforms

The self-reciprocal function of Theorem 1 was equivalent to the formulae (2.4) and (2.5). These formulae are limiting cases of the more general result*

$$\sum_{n\leqslant x}{}'\Lambda(n)n^{-s}=\frac{x^{1-s}}{1-s}-\frac{\zeta'}{\zeta}(s)+\sum_{n=1}^\infty\frac{x^{-2n-s}}{2n+s}-\sum_\rho\frac{x^{\rho-s}}{\rho-s}$$

$$(x>1;\ s\neq 1,\ -2n,\ \rho). \tag{4.1}$$

This suggests that there may be a corresponding extension of Theorem 1. However, we cannot prove such a result without some assumption. We need the result

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\Lambda(n)n^{-s}-\frac{N^{1-s}}{1-s}\right\}=-\frac{\zeta'}{\zeta}(s)\quad\{R(s)>\tfrac{1}{2}\}. \tag{4.2}$$

* E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 81.

If we assume the Riemann hypothesis, then (4.2) follows from (4.1), and an argument similar to the proof of Theorem 1 gives us

THEOREM 2. *If the Riemann hypothesis be true, and*

$$F_\alpha(x) = x^{\alpha-1}\left\{\sum_{n\leqslant x}{}' \Lambda(n)n^{-\alpha} - \frac{x^{1-\alpha}}{1-\alpha}\right\} + \frac{1}{2}\sum_\rho \frac{x^{\rho-1}}{\rho-\alpha} + A_\alpha(x),$$

where

$$A_\alpha(x) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1-\alpha} & (x<1), \\ \tfrac{1}{2}x^{\alpha-1}\left\{\log 2\pi + \tfrac{1}{2}\pi\tan\tfrac{1}{2}\alpha\pi - \dfrac{\Gamma'}{\Gamma}(\alpha)\right\} - \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\frac{x^{-2n-1}}{2n+\alpha} & (x>1), \end{cases}$$

then

$$F_\alpha(x) = T_\alpha\{F_\alpha(x)\} = \frac{1}{x}F_\alpha\left(\frac{1}{x}\right) + (2\alpha-1)x^{\alpha-1}\int_{1/x}^{\infty} t^{\alpha-1}F_\alpha(t)\,dt$$

for $0 < R(\alpha) < \frac{1}{2}$.

In the particular case $\alpha = \frac{1}{2}$ the transformation reduces to

$$T_{\frac{1}{2}}\{f(x)\} = \frac{1}{x}f\left(\frac{1}{x}\right),$$

and we can use (4.1) to show that

$$F_{\frac{1}{2}}(x) = \frac{1}{x}F_{\frac{1}{2}}\left(\frac{1}{x}\right)$$

without the Riemann hypothesis.

Now, by analogy with the result of (C) we would expect to find that

$$F_{1-\alpha}(x) = \frac{1}{x}F_\alpha\left(\frac{1}{x}\right)$$

or some similar result. We have

$$\begin{aligned} F_{1-\alpha}(x) - \frac{1}{x}F_\alpha\left(\frac{1}{x}\right) &= x^{-\alpha}\left\{\sum_{n\leqslant x}{}' \Lambda(n)n^{\alpha-1} - \frac{x^\alpha}{\alpha}\right\} + \frac{1}{2}\sum_\rho \frac{x^{\rho-1}}{\rho-1+\alpha} - \\ &\quad - x^{-\alpha}\left\{\sum_{n\leqslant 1/x}{}' \Lambda(n)n^{-\alpha} - \frac{x^{\alpha-1}}{1-\alpha}\right\} + \frac{1}{2}\sum_\rho \frac{x^{\rho-1}}{\rho-1+\alpha} + \\ &\quad + \begin{cases} \displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+\alpha} - \tfrac{1}{2}x^{-\alpha}\left\{\frac{\zeta'}{\zeta}(\alpha) + \frac{\zeta'}{\zeta}(1-\alpha)\right\} & (x<1), \\ -\displaystyle\sum_{n=0}^{\infty}\frac{x^{-2n-1}}{2n+1-\alpha} + \tfrac{1}{2}x^{-\alpha}\left\{\frac{\zeta'}{\zeta}(\alpha) + \frac{\zeta'}{\zeta}(1-\alpha)\right\} & (x>1). \end{cases} \end{aligned}$$

For $x < 1$ this becomes, by (4.1),

$$x^{-\alpha}\left[-\frac{x^{\alpha}}{\alpha}-\left\{\sum_{n\leqslant 1/x}{}' \Lambda(n)n^{-\alpha}-\frac{x^{\alpha-1}}{1-\alpha}\right\}-\sum_{\rho}\frac{x^{\alpha-\rho}}{\rho-\alpha}+\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{x^{2n+\alpha}}{2n+\alpha}+\frac{x^{\alpha}}{\alpha}-\frac{1}{2}\left\{\frac{\zeta'}{\zeta}(\alpha)+\frac{\zeta'}{\zeta}(1-\alpha)\right\}\right]$$
$$=\tfrac{1}{2}x^{-\alpha}\left\{\frac{\zeta'}{\zeta}(\alpha)-\frac{\zeta'}{\zeta}(1-\alpha)\right\}.$$

For $x > 1$ it becomes

$$x^{-\alpha}\left[\sum_{n\leqslant x}{}' \Lambda(n)n^{\alpha-1}-\frac{x^{\alpha}}{\alpha}+\sum_{\rho}\frac{x^{\rho-1+\alpha}}{\rho-1+\alpha}+\frac{x^{\alpha-1}}{1-\alpha}-\right.$$
$$\left.-\sum_{n=1}^{\infty}\frac{x^{-2n-1+\alpha}}{2n+1-\alpha}-\frac{x^{\alpha-1}}{1-\alpha}+\frac{1}{2}\left\{\frac{\zeta'}{\zeta}(\alpha)+\frac{\zeta'}{\zeta}(1-\alpha)\right\}\right]$$
$$=\tfrac{1}{2}x^{-\alpha}\left\{\frac{\zeta'}{\zeta}(\alpha)-\frac{\zeta'}{\zeta}(1-\alpha)\right\}.$$

Hence, for all $x > 0$

$$F_{1-\alpha}(x)-\frac{1}{x}F_{\alpha}\left(\frac{1}{x}\right)=\tfrac{1}{2}x^{-\alpha}\left\{\frac{\zeta'}{\zeta}(\alpha)-\frac{\zeta'}{\zeta}(1-\alpha)\right\}$$

or $$F_{1-\alpha}(x)+\tfrac{1}{2}x^{-\alpha}\frac{\zeta'}{\zeta}(1-\alpha)=\frac{1}{x}F_{\alpha}\left(\frac{1}{x}\right)+\tfrac{1}{2}x^{-\alpha}\frac{\zeta'}{\zeta}(\alpha).$$

Thus, if we put $$H_{\alpha}(x)=F_{\alpha}(x)+\tfrac{1}{2}x^{\alpha-1}\frac{\zeta'}{\zeta}(\alpha), \tag{4.3}$$

then we have

THEOREM 3. *If $\alpha \neq 1$, ρ, $-2n$, and $x > 0$, then*

$$H_{1-\alpha}(x)=\frac{1}{x}H_{\alpha}\left(\frac{1}{x}\right),$$

where $H_{\alpha}(x)$ is defined by (4.3).

We can now use the results of Theorems 2 and 3 to deduce a result corresponding to Theorem 2 for $\frac{1}{2} < R(\alpha) < 1$.

Now, for $R(\beta) < \frac{1}{2}$,

$$T_{\beta}\{x^{\beta-1}\}=x^{-\beta}+(2\beta-1)x^{\beta-1}\int_{1/x}^{\infty}t^{2\beta-2}\,dt=0,$$

and hence, on the Riemann hypothesis,

$$F_\beta(x) = \frac{1}{x}F_{1-\beta}\left(\frac{1}{x}\right)+\tfrac{1}{2}x^{\beta-1}\left\{\frac{\zeta'}{\zeta}(1-\beta)-\frac{\zeta'}{\zeta}(\beta)\right\}$$

$$= T_\beta\{F_\beta(x)\} = T_\beta\left\{\frac{1}{x}F_{1-\beta}\left(\frac{1}{x}\right)\right\}$$

$$= F_{1-\beta}(x)+(2\beta-1)x^{\beta-1}\int\limits_{1/x}^{\infty} u^{\beta-2}F_{1-\beta}\left(\frac{1}{u}\right)\,du$$

$$= F_{1-\beta}(x)+(2\beta-1)x^{\beta-1}\int\limits_{0}^{x} t^{-\beta}F_{1-\beta}(t)\,dt.$$

Hence,

$$F_{1-\beta}(x)+\tfrac{1}{2}x^{-\beta}\left\{\frac{\zeta'}{\zeta}(1-\beta)-\frac{\zeta'}{\zeta}(\beta)\right\}$$

$$= \frac{1}{x}F_{1-\beta}\left(\frac{1}{x}\right)+(2\beta-1)x^{-\beta}\int\limits_{0}^{1/x} t^{-\beta}F_{1-\beta}(t)\,dt$$

$$= T_{1-\beta}\{F_{1-\beta}(x)\}, \tag{4.4}$$

using the definition (2.3) for $T_{1-\beta}$. Also

$$T_{1-\beta}\{x^{-\beta}\} = x^{\beta-1}+(2\beta-1)x^{-\beta}\int\limits_{0}^{1/x} t^{-2\beta}\,dt = 0.$$

Hence, if we put

$$G_\alpha(x) = F_\alpha(x)+\tfrac{1}{2}x^{\alpha-1}\left\{\frac{\zeta'}{\zeta}(\alpha)-\frac{\zeta'}{\zeta}(1-\alpha)\right\}, \tag{4.5}$$

we have $$T_{1-\beta}\{G_{1-\beta}(x)\} = G_{1-\beta}(x).$$

That is

THEOREM 4. *If the Riemann hypothesis be true, and T_α and $G_\alpha(x)$ are defined by* (2.3) *and* (4.5), *then*

$$T_\alpha\{G_\alpha(x)\} = G_\alpha(x)$$

for $\frac{1}{2} < R(\alpha) < 1$ and $x > 0$.

INFINITE INTEGRALS INVOLVING STRUVE'S FUNCTIONS

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1. THE object of this note is to evaluate some infinite integrals involving Struve's functions, defined by the formula*

$$H_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})}.$$

It is supposed throughout that the constants a and b are *positive*, and the parameters ν, m, n, p are, for simplicity, supposed real; the formulae are, however, valid in suitably chosen complex domains of both constants and parameters.

Let

$$I = \int_0^\infty x^{p-1}e^{-ax^2}H_\nu(bx)\,dx \quad (\nu+p+1 > 0),$$

so that

$$I = \int_0^\infty x^{p-1}e^{-ax^2} \sum_{r=0}^{\infty} \frac{(-1)^r(bx)^{\nu+2r+1}}{2^{\nu+2r+1}\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})}\,dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r b^{\nu+2r+1}}{2^{\nu+2r+1}\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})} \int_0^\infty x^{\nu+2r+p}e^{-ax^2}\,dx,$$

the inversion of summation and integration being easily justifiable. Then

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r b^{\nu+2r+1}\Gamma(\frac{1}{2}\nu+r+\frac{1}{2}p+\frac{1}{2})}{2^{\nu+2r+2}\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})a^{\frac{1}{2}\nu+r+\frac{1}{2}p+\frac{1}{2}}}$$

$$= \frac{b^{\nu+1}\Gamma(\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2})}{2^{\nu+2}\Gamma(\frac{3}{2})\Gamma(\nu+\frac{3}{2})a^{\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2}}} \sum_0^\infty \frac{(-1)^r b^{2r}(\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2}, r)}{2^{2r}(\frac{3}{2}, r)(\nu+\frac{3}{2}, r)a^r},$$

where $\qquad (\alpha, r) \equiv \alpha(\alpha+1)(\alpha+2)...(\alpha+r-1), \quad (\alpha, 0) = 1.$

Thus we have, for $a > 0$, $\nu+p+1 > 0$,

$$\int_0^\infty x^{p-1}e^{-ax^2}H_\nu(bx)\,dx = \frac{b^{\nu+1}\Gamma(\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2})}{2^{\nu+1}\pi^{\frac{1}{2}}\Gamma(\nu+\frac{3}{2})a^{\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2}}}\, {}_2F_2\left(\begin{matrix}1, \frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2} \\ \frac{3}{2}, \nu+\frac{3}{2}\end{matrix}; -\frac{b^2}{4a}\right). \tag{1.1}$$

* G. N. Watson, *Theory of Bessel Functions*, Cambridge, 1922, § 10.4 (2).

Particular cases

(i) $p = 1$ (with $\nu > -2$):

$$\int_0^\infty e^{-ax^2} H_\nu(bx)\,dx = \frac{b^{\nu+1}\Gamma(\frac{1}{2}\nu+1)}{2^{\nu+1}\pi^{\frac{1}{2}}\Gamma(\nu+\frac{3}{2})a^{\frac{1}{2}\nu+1}}\,{}_2F_2\!\left(\begin{matrix}1, \frac{1}{2}\nu+1\\ \frac{3}{2}, \nu+\frac{3}{2}\end{matrix}; -\frac{b^2}{4a}\right). \tag{1.2}$$

(*a*) $\nu = 1$:

$$\int_0^\infty e^{-ax^2} H_1(bx)\,dx = \frac{b^2}{6\pi^{\frac{1}{2}}a^{\frac{3}{2}}}\,{}_1F_1\!\left(1, \tfrac{5}{2}; -\frac{b^2}{4a}\right). \tag{1.3}$$

(*b*) $\nu = -\frac{1}{2}$:

$$\int_0^\infty e^{-ax^2} H_{-\frac{1}{2}}(bx)\,dx = \sqrt{\left(\frac{b}{2\pi}\right)}\,\frac{\Gamma(\frac{3}{4})}{a^{\frac{3}{4}}}\,{}_1F_1\!\left(\tfrac{3}{4}; \tfrac{3}{2}; -\frac{b^2}{4a}\right). \tag{1.4}$$

(*c*) $\nu = \frac{1}{2}$:

$$\int_0^\infty \frac{e^{-ax^2}}{\sqrt{x}}(1-\cos bx)\,dx = \frac{b^2\Gamma(\frac{1}{4})}{16a^{\frac{5}{4}}}\,{}_2F_2\!\left(\begin{matrix}1, \frac{5}{4}\\ \frac{3}{2}, 2\end{matrix}; -\frac{b^2}{4a}\right). \tag{1.5}$$

(ii) $p = \nu+2$ (with $\nu > -\frac{3}{2}$):

$$\int_0^\infty x^{\nu+1}e^{-ax^2} H_\nu(bx)\,dx = \frac{b^{\nu+1}}{2^{\nu+1}\pi^{\frac{1}{2}}a^{\nu+\frac{3}{2}}}\,{}_1F_1\!\left(1; \tfrac{3}{2}; -\frac{b^2}{4a}\right). \tag{1.6}$$

In particular, if $\nu = \frac{1}{2}$,

$$\int_0^\infty xe^{-ax^2}(1-\cos bx)\,dx = \frac{b^2}{4a^2}\,{}_1F_1\!\left(1; \tfrac{3}{2}; -\frac{b^2}{4a}\right). \tag{1.7}$$

(iii) $p = 2-\nu$:

$$\int_0^\infty x^{1-\nu}e^{-ax^2} H_\nu(bx)\,dx = \frac{b^{\nu+1}}{2^{\nu+2}a^{\frac{3}{2}}\Gamma(\nu+\frac{3}{2})}\,{}_1F_1\!\left(1; \nu+\tfrac{3}{2}; -\frac{b^2}{4a}\right). \tag{1.8}$$

(*a*) $\nu = -\frac{1}{2}$:

$$\int_0^\infty x^{\frac{3}{2}}e^{-ax^2} H_{-\frac{1}{2}}(bx)\,dx = \frac{b^{\frac{1}{2}}}{(2a)^{\frac{3}{2}}}\exp\!\left(-\frac{b^2}{4a}\right). \tag{1.9}$$

(*b*) $\nu = \frac{1}{2}$:

$$\int_0^\infty e^{-ax^2}(1-\cos bx)\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\left\{1-\exp\!\left(-\frac{b^2}{4a}\right)\right\}, \tag{α}$$

which is a familiar result.

(iv) $\nu = \frac{1}{2}$ (with $p > -\frac{3}{2}$):

$$\int_0^\infty x^{p-\frac{3}{2}} e^{-ax^2}(1-\cos bx)\,dx = \frac{b^2\Gamma(\frac{3}{4}+\frac{1}{2}p)}{4a^{\frac{3}{4}+\frac{1}{2}p}}\,{}_2F_2\left(\begin{matrix}1, \frac{3}{4}+\frac{1}{2}p \\ \frac{3}{2}, 2\end{matrix}; -\frac{b^2}{4a}\right). \tag{1.10}$$

If we put $p = \frac{5}{2}$ and $p = \frac{3}{2}$ in this formula, of course, we merely get (1.7) and (α) again.

(v) $\nu = -\frac{1}{2}$ (with $p > -\frac{1}{2}$):

$$\int_0^\infty x^{p-1} e^{-ax^2} H_{-\frac{1}{2}}(bx)\,dx = \sqrt{\left(\frac{b}{2\pi}\right)}\,\frac{\Gamma(\frac{1}{4}+\frac{1}{2}p)}{a^{\frac{1}{4}+\frac{1}{2}p}}\,{}_1F_1\left(\tfrac{1}{4}+\tfrac{1}{2}p; \tfrac{3}{2}; -\frac{b^2}{4a}\right), \tag{1.11}$$

which, of course, reduces to (1.9) if we put $p = \frac{5}{2}$.

2. Following the same procedure, we get

$$\int_0^\infty x^{p-1} K_m(ax) H_n(bx)\,dx$$

$$= \frac{2^{p-1} b^{n+1}\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2}p+\frac{1}{2})\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2}p+\frac{1}{2})}{\pi^{\frac{1}{2}} a^{n+p-1}\Gamma(n+\frac{3}{2})} \times$$

$$\times\,{}_3F_2\left(\begin{matrix}1, \frac{1}{2}n+\frac{1}{2}m+\frac{1}{2}p+\frac{1}{2}, \frac{1}{2}n-\frac{1}{2}m+\frac{1}{2}p+\frac{1}{2} \\ \frac{3}{2}, n+\frac{3}{2}\end{matrix}; -\frac{b^2}{a^2}\right), \tag{2.1}$$

with $n+p > |m|-1$.

Particular cases

(i) $n = \frac{1}{2}$ (with $p > |m|-\frac{3}{2}$):

$$\int_0^\infty x^{p-\frac{3}{2}}(1-\cos bx) K_m(ax)\,dx = \frac{2^{p-\frac{5}{2}} b^2\Gamma(\frac{3}{4}+\frac{1}{2}m+\frac{1}{2}p)\Gamma(\frac{3}{4}-\frac{1}{2}m+\frac{1}{2}p)}{a^{p+\frac{3}{2}}} \times$$

$$\times\,{}_3F_2\left(\begin{matrix}1, \frac{3}{4}+\frac{1}{2}m+\frac{1}{2}p, \frac{3}{4}-\frac{1}{2}m+\frac{1}{2}p \\ \frac{3}{2}, 2\end{matrix}; -\frac{b^2}{a^2}\right). \tag{2.2}$$

(a) $m = \frac{1}{2}$ (with $p > -1$):

$$\int_0^\infty x^{p-2} e^{ax}(1-\cos bx)\,dx = \frac{b^2\Gamma(1+p)}{2a^{p+1}}\left({}_3F_2\begin{matrix}1, 1+\frac{1}{2}p, \frac{1}{2}+\frac{1}{2}p \\ \frac{3}{2}, 2\end{matrix}; -\frac{b^2}{a^2}\right). \tag{2.3}$$

In particular, taking $p = 3$, $p = 1$, and $p = 2$, we get the elementary results

$$\int_0^\infty xe^{-ax}(1-\cos bx)\,dx = \frac{3b^2}{a^4}\,F\left(1, \tfrac{5}{2}; \tfrac{3}{2}; -\frac{b^2}{a^2}\right), \tag{2.4}$$

$$\int_0^\infty \frac{e^{-ax}}{x}(1-\cos bx)\,dx = \frac{b^2}{2a^2}\,F\left(1, 1; 2; -\frac{b^2}{a^2}\right), \tag{2.5}$$

$$\int_0^\infty e^{-ax}(1-\cos bx)\,dx = \frac{b^2}{a(a^2+b^2)}. \tag{β}$$

(*b*) $p = \frac{3}{2}-m$ (with $m < \frac{3}{2}$):

$$\int_0^\infty \frac{1-\cos bx}{x^m}K_m(ax)\,dx = \frac{b^2\pi^{\frac{1}{2}}\Gamma(\frac{3}{2}-m)}{2^{m+1}a^{3-m}}\,F\left(1, \tfrac{3}{2}-m; 2; -\frac{b^2}{a^2}\right). \tag{2.6}$$

If we put $m = \frac{1}{2}$ and $m = -\frac{1}{2}$, we merely get (2.5) and (β) again.

(*c*) $p = \frac{3}{2}+m$ (with $m > -\frac{3}{2}$):

$$\int_0^\infty x^m(1-\cos bx)K_m(ax)\,dx = \frac{2^{m-1}b^2\pi^{\frac{1}{2}}}{a^{m+3}}\,\Gamma(\tfrac{3}{2}+m)\,F\left(1, \tfrac{3}{2}+m; 2; -\frac{b^2}{a^2}\right). \tag{2.7}$$

Since $K_m = K_{-m}$, this is really the same formula as (2.6).

(*d*) $p = \frac{5}{2}-m$ (with $m < 2$):

$$\int_0^\infty \frac{1-\cos bx}{x^{m-1}}K_m(ax)\,dx = \frac{b^2\Gamma(2-m)}{2^{m-1}a^{4-m}}\,F\left(1, 2-m; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \tag{2.8}$$

(*e*) $p = \frac{5}{2}+m$ (with $m > -2$):

$$\int_0^\infty x^{m+1}(1-\cos bx)K_m(ax)\,dx = \frac{2^{m+1}b^2\Gamma(m+2)}{a^{m+4}}\,F\left(1, m+2; \tfrac{3}{2}; -\frac{b^2}{a^2}\right), \tag{2.9}$$

which is really the same as (2.8).

(ii) $m = \frac{1}{2}$ (with $n+p > -\frac{1}{2}$):

$$\int_0^\infty x^{p-\frac{1}{2}}e^{-ax}H_n(bx)\,dx = \frac{2^{p-\frac{1}{2}}b^{n+1}\Gamma(\frac{3}{4}+\frac{1}{2}n+\frac{1}{2}p)\Gamma(\frac{1}{4}+\frac{1}{2}n+\frac{1}{2}p)}{\pi a^{n+p+\frac{1}{2}}\Gamma(n+\frac{3}{2})}\times$$

$$\times\,{}_3F_2\left(\begin{matrix}1, \frac{3}{4}+\frac{1}{2}n+\frac{1}{2}p, \frac{1}{4}+\frac{1}{2}n+\frac{1}{2}p \\ \frac{3}{2}, n+\frac{3}{2}\end{matrix}; -\frac{b^2}{a^2}\right). \tag{2.10}$$

For $n = \frac{1}{2}$ this formula is the same as (2.3).

(*a*) $p = \frac{3}{2} - n$:

$$\int_0^\infty \frac{e^{-ax}}{x^n} H_n(bx)\,dx = \frac{b^{n+1}}{2^n \pi^{\frac{1}{2}} a^2 \Gamma(n+\frac{3}{2})} F\left(1, 1; n+\frac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.11)$$

For $n = -\frac{1}{2}$, this formula becomes

$$\int_0^\infty x^{\frac{1}{2}} e^{-ax} H_{-\frac{1}{2}}(bx)\,dx = \sqrt{\left(\frac{2b}{\pi}\right)} \frac{1}{a^2+b^2}. \quad (2.12)$$

(*b*) $p = \frac{3}{2} + n$ (with $n > -1$):

$$\int_0^\infty x^n e^{-ax} H_n(bx)\,dx = \frac{(2b)^{n+1}\Gamma(n+1)}{\pi a^{2n+2}} F\left(1, n+1; \frac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.13)$$

A particular case is

$$\int_0^\infty x e^{-ax} H_1(bx)\,dx = \frac{4b^2}{\pi a^4} F\left(1, 2; \frac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.14)$$

(*c*) $p = \frac{5}{2} - n$:

$$\int_0^\infty \frac{e^{-ax}}{x^{n-1}} H_n(bx)\,dx = \frac{b^{n+1}}{2^{n-1}\pi^{\frac{1}{2}}\Gamma(n+\frac{3}{2})a^3} F\left(1, 2; n+\frac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.15)$$

When $n = -\frac{1}{2}$, this becomes

$$\int_0^\infty x^{\frac{3}{2}} e^{-ax} H_{-\frac{1}{2}}(bx)\,dx = \frac{2^{\frac{3}{2}} a\sqrt{b}}{\pi^{\frac{1}{2}}(a^2+b^2)^2}. \quad (2.16)$$

(*d*) $p = \frac{5}{2} + n$ (with $n > -\frac{3}{2}$):

$$\int_0^\infty x^{n+1} e^{-ax} H_n(bx)\,dx = \frac{2^{n+2} b^{n+1}\Gamma(n+2)}{\pi a^{2n+3}} F\left(1, n+2; \frac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.17)$$

(*e*) $n = -\frac{1}{2}$ (with $p > 0$):

$$\int_0^\infty x^{p-\frac{3}{2}} e^{-ax} H_{-\frac{1}{2}}(bx)\,dx = \sqrt{\left(\frac{2b}{\pi}\right)} \frac{\Gamma(p)}{a^p} F\left(\frac{1}{2}p, \frac{1}{2}+\frac{1}{2}p; \frac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.18)$$

For $p = 2$, this becomes (2.12). For $p = \frac{3}{2}$, we get

$$\int_0^\infty e^{-ax} H_{-\frac{1}{2}}(bx)\,dx = \sqrt{\left(\frac{b}{2a^3}\right)} F\left(\frac{3}{4}, \frac{5}{4}; \frac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.19)$$

(iii) $p = 1$ (with $n > |m|-2$):

$$\int_0^\infty K_m(ax)H_n(bx)\,dx = \frac{b^{n+1}\Gamma(\frac{1}{2}n+\frac{1}{2}m+1)\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)}{a^{n+2}\Gamma(n+\frac{3}{2})\sqrt{\pi}}\times$$

$$\times\,{}_3F_2\!\left(\begin{matrix}1, \frac{1}{2}n+\frac{1}{2}m+1, \frac{1}{2}n-\frac{1}{2}m+1\\ \frac{3}{2}, n+\frac{3}{2}\end{matrix}; -\frac{b^2}{a^2}\right). \quad (2.20)$$

(*a*) $m = \pm\frac{1}{2}$ (with $n > -\frac{3}{2}$):

$$\int_0^\infty \frac{e^{-ax}}{\sqrt{x}} H_n(bx)\,dx = \frac{b^{n+1}}{2^n a^{n+\frac{3}{2}}\sqrt{\pi}}\,{}_3F_2\!\left(\begin{matrix}1, \frac{5}{4}+\frac{1}{2}n, \frac{3}{4}+\frac{1}{2}n\\ \frac{3}{2}, n+\frac{3}{2}\end{matrix}; -\frac{b^2}{a^2}\right). \quad (2.21)$$

When $n = -\frac{1}{2}$, this becomes

$$\int_0^\infty \frac{e^{-ax}}{\sqrt{x}} H_{-\frac{1}{2}}(bx)\,dx = \frac{1}{a}\sqrt{\left(\frac{2b}{\pi}\right)}\,F\!\left(1, \tfrac{1}{2}; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.22)$$

For $n = \frac{3}{2}$, (2.21) reduces to

$$\int_0^\infty \frac{e^{-ax}}{\sqrt{x}} H_{\frac{3}{2}}(bx)\,dx = \frac{b^{\frac{5}{2}}}{2^{\frac{3}{2}}a^3\sqrt{\pi}}\,F\!\left(1, 2; 3; -\frac{b^2}{a^2}\right). \quad (2.23)$$

(*b*) $m = 1+n$ (with $n > -\frac{3}{2}$):

$$\int_0^\infty K_{n+1}(ax)H_n(bx)\,dx = \frac{b^{n+1}}{a^{n+2}}\,F\!\left(1, \tfrac{1}{2}; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.24)$$

For $n = \frac{1}{2}$, this becomes

$$\int_0^\infty K_{\frac{3}{2}}(ax)(1-\cos bx)\,dx = \sqrt{\left(\frac{\pi b^4}{2a^5}\right)}\,F\!\left(1, \tfrac{1}{2}, \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.25)$$

(*c*) $m = 1-n$ (with $n > -\frac{1}{2}$):

$$\int_0^\infty K_{1-n}(ax)H_n(bx)\,dx = \frac{b^{n+1}}{a^{n+2}(2n+1)}\,F\!\left(1, n+\tfrac{1}{2}, n+\tfrac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.26)$$

(*d*) $n = -\frac{1}{2}$ (with $|m| < \frac{3}{2}$):

$$\int_0^\infty K_m(ax)H_{-\frac{1}{2}}(bx)\,dx$$

$$= \frac{b^{\frac{1}{2}}\Gamma(\frac{3}{4}+\frac{1}{2}m)\Gamma(\frac{3}{4}-\frac{1}{2}m)}{a^{\frac{3}{2}}\pi^{\frac{1}{2}}}\,F\!\left(\tfrac{3}{4}+\tfrac{1}{2}m, \tfrac{3}{4}-\tfrac{1}{2}m; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \quad (2.27)$$

(iv) $p = 2-m-n$ (with $m < \frac{3}{2}$):

$$\int_0^\infty \frac{K_m(ax)H_n(bx)}{x^{m+n-1}}\,dx = \frac{b^{n+1}\Gamma(\frac{3}{2}-m)a^{m-3}}{2^{m+n}\Gamma(\frac{3}{2}+n)}\,F\left(1, \tfrac{3}{2}-m; \tfrac{3}{2}+n; -\frac{b^2}{a^2}\right). \tag{2.28}$$

(*a*) $m = -n$ (with $n > -\frac{3}{2}$):

$$\int_0^\infty xK_{-n}(ax)H_n(bx)\,dx = \frac{b^{n+1}}{a^{n+1}(a^2+b^2)}. \tag{2.29}$$

(*b*) $n = -\frac{1}{2}$ (with $m < \frac{3}{2}$):

$$\int_0^\infty \frac{K_m(ax)}{x^{m-\frac{3}{2}}}H_{-\frac{1}{2}}(bx)\,dx = \frac{b^{\frac{1}{2}}\Gamma(\frac{3}{2}-m)}{2^{m-\frac{1}{2}}a^m}(a^2+b^2)^{m-\frac{3}{2}}. \tag{2.30}$$

(v) $p = 2+m-n$ (with $m > -\frac{3}{2}$):

$$\int_0^\infty x^{m-n+1}K_m(ax)H_n(bx)\,dx = \frac{2^{m-n}b^{n+1}\Gamma(m+\frac{3}{2})}{a^{m+3}\Gamma(n+\frac{3}{2})}\times$$

$$\times F\left(1, m+\tfrac{3}{2}; n+\tfrac{3}{2}; -\frac{b^2}{a^2}\right). \tag{2.31}$$

(vi) $p = 2-m+n$ (with $n > m-\frac{3}{2}$ and $n > -\frac{3}{2}$):

$$\int_0^\infty x^{n-m+1}K_m(ax)H_n(bx)\,dx$$

$$= \frac{2^{n-m+1}b^{n+1}\Gamma(\frac{3}{2}-m+n)}{a^{2n-m+3}\sqrt{\pi}}\,F\left(1, \tfrac{3}{2}-m+n; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \tag{2.32}$$

For $m = -\frac{1}{2}$ (with $n > -\frac{3}{2}$) we get

$$\int_0^\infty x^{n+1}e^{-ax}H_n(bx)\,dx = \frac{2^{n+2}b^{n+1}\Gamma(n+2)}{a^{2n+3}\pi}\,F\left(1, n+2; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \tag{2.33}$$

(vii) $p = 2+m+n$ (with $n > -\frac{3}{2}$ and $n > -\frac{3}{2}-m$):

$$\int_0^\infty x^{m+n+1}K_m(ax)H_n(bx)\,dx$$

$$= \frac{2^{m+n+1}b^{n+1}\Gamma(\frac{3}{2}+m+n)}{\pi^{\frac{1}{2}}a^{m+2n+3}}\,F\left(1, \tfrac{3}{2}+m+n; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \tag{2.34}$$

For $n = -\frac{1}{2}$ and $m > -1$ we get

$$\int\limits_0^\infty x^{m+\frac{1}{2}} K_m(ax) H_{-\frac{1}{2}}(bx)\, dx$$

$$= \frac{2^m \Gamma(m+1)}{a^{m+2}} \sqrt{\left(\frac{2b}{\pi}\right)}\, F\left(1, 1+m; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \qquad (2.35)$$

(viii) $n = -\frac{1}{2}$ (with $p+\frac{1}{2} > |m|$):

$$\int\limits_0^\infty x^{p-1} K_m(ax) H_{-\frac{1}{2}}(bx)\, dx = \frac{2^{p-1} b^{\frac{1}{2}} \Gamma(\frac{1}{4}+\frac{1}{2}m+\frac{1}{2}p) \Gamma(\frac{1}{4}-\frac{1}{2}m+\frac{1}{2}p)}{\pi^{\frac{1}{2}} a^{p+\frac{1}{2}}} \times$$

$$\times F\left(\tfrac{1}{4}+\tfrac{1}{2}m+\tfrac{1}{2}p, \tfrac{1}{4}-\tfrac{1}{2}m+\tfrac{1}{2}p; \tfrac{3}{2}; -\frac{b^2}{a^2}\right). \qquad (2.36)$$

Particular cases of this formula have already occurred above.

A MEAN VALUE THEOREM CONCERNING FAREY SERIES

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THIS paper relates to the *Farey series* of order $n > 1$, that is, the ascending series $\mathfrak{F}_n$ of irreducible fractions between $\frac{0}{1}$ and $\frac{1}{1}$ (inclusive) whose denominators do not exceed n. If

$$\frac{a_0}{b_0}, \quad \frac{a_1}{b_1}, \quad \frac{a_2}{b_2}$$

are the reduced forms of three consecutive terms t_0, t_1, t_2 in $\mathfrak{F}_n$, it is familiar that*

$$\frac{a_1}{b_1} = \frac{a_0+a_2}{b_0+b_2} \tag{0.1}$$

and

$$a_1 b_0 - a_0 b_1 = 1. \tag{0.2}$$

The first of these fundamental properties suggests that we should find the position of the middle term $(a_0+a_2)/(b_0+b_2)$, called the *mediant* of a_0/b_0 and a_2/b_2, among the usual mean values of these neighbouring terms; but there is evidence that neither the arithmetic mean nor the harmonic mean is *comparable*† with the mediant. For example, the first half of $\mathfrak{F}_{10}$ is

$$\tfrac{0}{1}, \tfrac{1}{10}, \tfrac{1}{9}, \tfrac{1}{8}, \tfrac{1}{7}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{2}{9}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{3}{10}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{4}{9}, \tfrac{1}{2}.$$

Here $\frac{3}{10}$ is less than the harmonic mean (with unit weights) of the adjacent fractions,‡ while, e.g., $\frac{3}{7}$ is greater than the respective arithmetic mean, and no simple law is obvious that would allow us to foresee when the one or the other of these possibilities will happen.§ Yet, once any particular $\mathfrak{F}_n$ has been constructed, it is easy

* See e.g. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford, 1938), chap. 3.

† If understood as in G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities* (Cambridge, 1934), 5.

‡ This occurs for the first time in $\mathfrak{F}_{10}$ if t_0, t_1, t_2 are to be in the interior of the first half of $\mathfrak{F}_n$.

§ Apart from rules like the following, where $\mathfrak{A}$ denotes the arithmetic mean of the neighbours of the term in question, and $\mathfrak{H}$ the harmonic mean:

(i) if $t_1 \begin{cases} > \mathfrak{A}, \\ \leqslant \mathfrak{H}, \end{cases}$ then $(1-t_1) \begin{cases} \leqslant \mathfrak{H}, \\ > \mathfrak{A}; \end{cases}$

(ii) if $t_2 = \frac{1}{2}$, then $t_1 < \mathfrak{H}$ $(n > 6)$;

(iii) if $t_1 = \frac{1}{2}$, then $t_1 = \mathfrak{A}$;

(iv) if $t_2 = 1$, then $t_1 < \mathfrak{H}$ $(n > 3)$.

to discern both cases; for t_1 is not greater than the harmonic mean if $a_0 \geqslant a_2$, and not less than the arithmetic mean if $b_0 \leqslant b_2$ (Lemmas 2 and 3).

We can prove that there is no other alternative. Hence no term of any Farey series falls between the arithmetic and the harmonic means of its neighbours (Theorem 5), and we shall be able to improve slightly upon this main result.*

1. The relation of the mediant to the arithmetic, geometric, and harmonic means

When trying to establish relations between the term t_1 of $\mathfrak{F}_n$ and the familiar mean values of the neighbours t_0, t_2, one will, perhaps, notice first that t_1 never equals the geometric mean.

The denominators of adjacent fractions in $\mathfrak{F}_n$ are coprime. In particular

$$(b_0, b_1) = 1, \qquad (b_1, b_2) = 1,$$

and therefore

$$(b_1^2, b_0 b_2) = 1.$$

Hence, in view of (0.1), we have the alternative

$$\frac{a_0+a_2}{b_0+b_2} \lessgtr \sqrt{\left(\frac{a_0 a_2}{b_0 b_2}\right)}, \tag{1.1}$$

i.e., for $a_0 > 0$,

$$\frac{a_0}{a_2}+\frac{a_2}{a_0} \lessgtr \frac{b_0}{b_2}+\frac{b_2}{b_0}. \tag{1.2}$$

Observing that $f(x) = x+1/x$ is decreasing for $0 < x \leqslant 1$, and increasing for $x \geqslant 1$, and that

$$\frac{a_0}{a_2} < \frac{b_0}{b_2}, \tag{1.3}$$

we may deduce from the fact that (1.1) and (1.2) are equivalent for $a_0 > 0$ the following lemma:

LEMMA 1. *If either $a_0/a_2 \geqslant 1$, or $b_0/b_2 \leqslant 1$, then the mediant is, respectively, less or greater than the geometric mean.*

This holds also for $a_0 = 0$.

* I wish to thank here Professor G. H. Hardy, who read an earlier version of this paper, for remarks which led to substantial improvements. In particular, he suggested attending explicitly to the geometric mean (see the beginning of § 1), and conjectured Theorem 4, a sweeping generalization of the preceding theorems.

We can prove considerably more by comparing now the mediant with the arithmetic and harmonic means. We write*

$$\mathfrak{M}_r(t_0, t_2) = \{\tfrac{1}{2}(t_0^r+t_2^r)\}^{1/r}.$$

$\mathfrak{M}_r$ reduces to the arithmetic mean and the harmonic mean for $r = 1$ and -1, respectively. If $t_0 t_2 = 0$, then $\mathfrak{M}_{-1}$ is to be defined as zero.

Consider first the difference

$$A = \frac{a_0+a_2}{b_0+b_2} - \mathfrak{M}_1\left(\frac{a_0}{b_0}, \frac{a_2}{b_2}\right)$$

between mediant and arithmetic mean. We obtain

$$A = \frac{a_0+a_2}{b_0+b_2} - \frac{1}{2}\left(\frac{a_0}{b_0}+\frac{a_2}{b_2}\right) = \frac{a_2 b_0 - a_0 b_2}{2b_0 b_2(b_0+b_2)}(b_2-b_0). \tag{1.4}$$

Thus we have, with regard to (1.3),

LEMMA 2. *If* $b_0 \lesseqgtr b_2$, *then* $A \gtreqless 0$.

There is no difficulty in finding out when $A = 0$, for this is equivalent to $b_0 = b_2$. In that case $a_2 = a_0+\epsilon$, where $\epsilon = 1$ or 2; otherwise there would be more than one term of $\mathfrak{F}_n$ between a_0/b_0 and $a_2/b_2 = (a_0+\epsilon)/b_0$. Hence, by (0.1),

$$a_1 = (2a_0+\epsilon)/\lambda, \qquad b_1 = 2b_0/\lambda,$$

where λ is a positive integer, and so, by (0.2),

$$b_1 = 2/\epsilon = 1 \text{ or } 2.$$

The alternative $b_1 = 1$ can be ruled out, and we find $a_1/b_1 = \frac{1}{2}$.

It is now to be recalled that along with a/b its complement $(b-a)/b$ to unity is a term of $\mathfrak{F}_n$; for both have the same denominator and, if the one is an irreducible vulgar fraction, so is the other. Hence the neighbours of $\frac{1}{2}$ in $\mathfrak{F}_n$ are complements to one another, and thus $\frac{1}{2}$ is their arithmetic mean. This completes the proof that $A = 0$ and $a_1/b_1 = \frac{1}{2}$ are equivalent.

Consider next the difference

$$H = \frac{a_0+a_2}{b_0+b_2} - \mathfrak{M}_{-1}\left(\frac{a_0}{b_0}, \frac{a_2}{b_2}\right)$$

between mediant and harmonic mean, and assume, preliminarily, $a_0/b_0 > 0$. Then

$$H = \frac{a_0+a_2}{b_0+b_2} - 2\bigg/\left(\frac{b_0}{a_0}+\frac{b_2}{a_2}\right),$$

* In accordance with *Inequalities* (op. cit.).

i.e.
$$H = \frac{a_2 b_0 - a_0 b_2}{(a_0 b_2 + a_2 b_0)(b_0+b_2)}(a_2 - a_0). \tag{1.5}$$

The latter holds also if $a_0/b_0 = \frac{0}{1}$.

Taking (1.3) once more into account, we obtain

LEMMA 3. *If* $a_0 \lesseqgtr a_2$, *then* $H \gtreqless 0$.

Since $\mathfrak{M}_{-1}(t_0, t_2) < \mathfrak{M}_1(t_0, t_2)$, we can combine Lemmas 2 and 3 into

LEMMA 4. *If* $(a_2-a_0)(b_2-b_0) \gtreqless 0$, *then* $(a_0+a_2)/(b_0+b_2)$ *lies, respectively, outside the interval* $(\mathfrak{M}_{-1}, \mathfrak{M}_1)$, *or at one end, or inside.*

The case $a_1/b_1 = \frac{1}{2}$ is trivial, and we may ignore it for the moment. If we then represent the numbers, in the usual way, by points on a straight line, the point corresponding to the mediant divides the segment $\mathfrak{M}_{-1}\mathfrak{M}_1$ in the ratio H/A. It follows from (1.4) and (1.5) that

$$\frac{H}{A} = \frac{a_2-a_0}{b_2-b_0}\mathfrak{M}_1^{-1}\left(\frac{a_0}{b_0}, \frac{a_2}{b_2}\right),$$

which amends the merely qualitative Lemma 4.

We shall prove in § 3 that always

$$(a_2-a_0)(b_2-b_0) \geqslant 0, \tag{1.6}$$

and this will appear to be a matter of elementary number theory, while so far we have made little use of the fact that the a's and b's are integers.

2. A note on inequalities

It seems worth observing that among the implications of our lemmas one—which has, however, no bearing upon Farey series—may be deduced from tolerably well-known inequalities. Let us suppose for the following digression that

$$(a_2-a_0)(b_2-b_0) \leqslant 0. \tag{2.1}$$

If so, Tchebychef's inequality* is essentially

$$\mathfrak{M}_r(a_0, a_2)\,\mathfrak{M}_r\left(\frac{1}{b_0}, \frac{1}{b_2}\right) < \mathfrak{M}_r\left(\frac{a_0}{b_0}, \frac{a_2}{b_2}\right) \quad (r > 0),$$

unless the a's or the b's are equal. Therefore, under the same condition,

$$\frac{\mathfrak{M}_r(a_0, a_2)}{\mathfrak{M}_{-r}(b_0, b_2)} < \mathfrak{M}_r\left(\frac{a_0}{b_0}, \frac{a_2}{b_2}\right) \quad (r > 0).$$

* *Inequalities*, 43.

Next,* $$\mathfrak{M}_s(b_0, b_2) < \mathfrak{M}_r(b_0, b_2) \quad (s < r;\ r > 0), \tag{2.2}$$

and so $$\mathfrak{M}_{-r}(b_0, b_2) < \mathfrak{M}_r(b_0, b_2),$$

unless the b's are equal. Hence we find that

$$\frac{\mathfrak{M}_r(a_0, a_2)}{\mathfrak{M}_r(b_0, b_2)} < \mathfrak{M}_r\left(\frac{a_0}{b_0}, \frac{a_2}{b_2}\right) \quad (r > 0), \tag{2.3}$$

unless the b's are equal.

Finally, take $r = 1$. Then (2.3) shows that the mediant is less than the arithmetic mean, if $b_0 \neq b_2$. The same would follow from Lemmas 2 and 4; if (2.1) were true, the mediant would lie in the interval between the arithmetic and harmonic means, the latter included. Here again $b_0 \neq b_2$ is to be assumed.

3. Numerators and denominators of neighbours, second neighbours, etc. in Farey series

As a preliminary to (1.6) we prove the simpler

THEOREM 1. *The numerators and denominators of neighbours in $\mathfrak{F}_n$ are similarly ordered.*

We call them *similarly ordered*† if

$$(a_1 - a_0)(b_1 - b_0) \geqslant 0. \tag{3.1}$$

Our first proof is geometrical. We represent the terms a/b of $\mathfrak{F}_n$ by points (b, a) of the integral lattice.‡ Geometrically, (0.2) means that $\frac{1}{2}$ is the area of the triangle whose base joins (b_0, a_0) and (b_1, a_1), the origin O being the apex.

The base, being a distance between lattice points, is not less than unity, and so the altitude does not exceed unity. Therefore the straight line through (b_0, a_0) and (b_1, a_1) contains at least one point P at a distance from O not exceeding unity. Plainly, (b_1, a_1) is one of the points Q inside or on the boundary of the triangle defined by the lines $y = 1$, $y = x$, $x = n$. After applying the transformation $x \to x - 1$, $y \to y - 1$, the coordinates of Q are still non-negative, those of P non-positive. Hence any two points (x_0, y_0), (x_1, y_1) on the line PQ satisfy $(y_1 - y_0)(x_1 - x_0) \geqslant 0$, of which (3.1) is a particular case.

As an alternative, we give an indirect proof of (3.1). From

$$(a_1 - a_0)(b_1 - b_0) < 0$$

* *Inequalities*, 26. † Ibid. 43. ‡ *Theory of Numbers* (op. cit.).

it follows, in view of $a_0/b_0 < a_1/b_1$, that $a_0 < a_1$ and $b_0 > b_1$, and thus

$$\frac{a_0}{b_0} < \frac{a_1}{b_0} < \frac{a_1}{b_1}$$

(and also $a_0/b_0 < a_0/b_1 < a_1/b_1$). Hence, unless (3.1) is true, there is at least one term of $\mathfrak{F}_n$ between a_0/b_0 and a_1/b_1.

It is convenient to call *second neighbours*, *third neighbours*, and so on, in a series such terms as are separated by one term or two terms, etc.

We come now to

THEOREM 2. *The numerators and denominators of second neighbours in $\mathfrak{F}_n$ are similarly ordered.*

We prove it by a natural development of the second proof of Theorem 1. Suppose that

$$(a_m - a_0)(b_m - b_0) < 0, \tag{3.2}$$

where a_m and b_m are the numerator and denominator of the mth term after a_0/b_0 in $\mathfrak{F}_n$. Theorem 2 will be established if we show that this requires $m > 2$.

From (3.2) and $a_0/b_0 < a_m/b_m$ it follows at once that

$$a_0 < a_m, \qquad b_0 > b_m. \tag{3.3}$$

Thus $b_0 > 1$, and so $a_0 > 0$. Therefore $a_m > 1$ and $a_m/b_m < \frac{1}{1}$.

We may write, instead of (3.3),

$$a_0 + 1 \leqslant a_m, \qquad b_0 - 1 \geqslant b_m, \tag{3.4}$$

and so

$$\frac{a_0}{b_0} < \frac{a_0}{b_0 - 1} < \frac{a_0 + 1}{b_0} < \frac{a_0 + 1}{b_0 - 1} \leqslant \frac{a_m}{b_m}. \tag{3.5}$$

The second '$<$' is an immediate deduction from $(a_0+1)/b_0 < 1$. It is now obvious that $m \geqslant 3$.

We can, however, go a little further and determine when $m = 3$. In this case we deduce from (3.4) and (3.5) that $a_m = a_0 + 1$ and $b_m = b_0 - 1$. At the same time, (3.5) might contain a_1/b_1 and a_2/b_2 in reducible form: say

$$a_1 = a_0/\lambda, \qquad a_2 = (a_0+1)/\mu,$$
$$b_1 = (b_0 - 1)/\lambda, \qquad b_2 = b_0/\mu,$$

where λ, μ are positive integers. Applying (0.2) to the neighbouring terms a_0/b_0, a_1/b_1 and a_2/b_2, a_3/b_3, we find $\lambda = a_0$ and $\mu = a_0 + 1$, and so

$$a_1 = 1, \qquad a_2 = 1,$$
$$b_1 = (b_0 - 1)/a_0, \qquad b_2 = b_0/(a_0+1).$$

Hence, again by (0.2), we obtain $b_0 = (a_0+1)^2$, so that a_0/b_0 and the following terms of $\mathfrak{F}_n$ become

$$\frac{a_0}{(a_0+1)^2},\quad \frac{1}{a_0+2},\quad \frac{1}{a_0+1},\quad \frac{a_0+1}{a_0(a_0+2)}. \tag{3.6}$$

Here the largest denominator is $(a_0+1)^2$. Unless

$$(a_0+1)^2 < 2a_0+3, \tag{3.7}$$

(0.1) shows that $\mathfrak{F}_n$ contains the fraction $2/(2a_0+3)$ between $1/(a_0+2)$ and $1/(a_0+1)$. The positive solution of (3.7) (for λ was defined as a positive integer, and $a_0 = \lambda$) is $a_0 = 1$, and (3.6) takes the form

$$\tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3}.$$

These are successive terms in $\mathfrak{F}_4$ only. We have thus proved

Theorem 3. *The numerators and denominators of third neighbours in $\mathfrak{F}_n$ are similarly ordered, except for $n = 4$.*

We shall make no use of this theorem: we merely wanted to show what can be proved by a slight refinement of the argument. Even more is true, in fact, namely

Theorem 4. *Given k, there is a number $N(k)$ such that the numerators and denominators of k-th neighbours in $\mathfrak{F}_n$ are similarly ordered, if only $n \geqslant N$.*

A proof will be given in another paper.

4. A mean-value theorem and its extension

Combining Theorem 2 with Lemma 4 and the discussion of the case $A = 0$, we obtain

Theorem 5. *If t_0, t_1, t_2 are successive terms in any Farey series, then either $t_1 \leqslant \mathfrak{M}_{-1}(t_0, t_2)$, or $t_1 \geqslant \mathfrak{M}_1(t_0, t_2)$. In the second case equality holds for $t_1 = \frac{1}{2}$ only.*

There are companions of this theorem which relate to the *Farey dissection,** that is, the set $\mathfrak{D}_n$ of mediants of the neighbours in $\mathfrak{F}_n$.

In Lemma 4 we replace the suffix 2 by 1. In view of Theorem 1 we then have

Theorem 6. *No term of the Farey dissection $\mathfrak{D}_n$ lies between the harmonic and the arithmetic means of those fractions in $\mathfrak{F}_n$ of which it is the mediant.*

* *Theory of Numbers* (op. cit.)

We come now to a converse of Theorem 6. Let t_{01} denote the mediant of t_0, t_1; and t_{12} the mediant of t_1, t_2. It is plain that $(a_0+a_1)/(b_0+b_1)$ is already the irreducible form of t_{01}. Otherwise the denominator of the reduced form would be at most $\frac{1}{2}(b_0+b_1) < n$, so that t_{01} would occur in $\mathfrak{F}_n$. Similarly, $t_{12} = (a_1+a_2)/(b_1+b_2)$ is irreducible. We have, therefore, without ambiguity, as mediant of t_{01} and t_{12} the fraction

$$\frac{a_0+2a_1+a_2}{b_0+2b_1+b_2}.$$

According to (0.1) we may write $a_0+a_2 = \lambda a_1$, where λ is a positive integer, and $b_0+b_2 = \lambda b_1$. Hence

$$\frac{a_0+2a_1+a_2}{b_0+2b_1+b_2} = \frac{a_1}{b_1} = t_1. \tag{4.1}$$

In Lemma 4 now replace a_0, b_0 by (a_0+a_1), (b_0+b_1) and a_2, b_2 by (a_1+a_2), (b_1+b_2). This leaves $(a_2-a_0)(b_2-b_0)$ unaltered. By appealing to Theorem 2 and (4.1), we arrive at

THEOREM 7. *No term of $\mathfrak{F}_n$ lies between the harmonic and the arithmetic means of the adjacent fractions of $\mathfrak{D}_n$.*

By Theorem 5 the term t_1 is excluded from the interval

$$\mathfrak{M}_{-1}(t_0, t_2) < x < \mathfrak{M}_1(t_0, t_2). \tag{4.2}$$

Except for the trivial case $t_1 = \frac{1}{2}$, even the upper bound of the interval (4.2) is inaccessible to t_1 (while in $\mathfrak{F}_n$, for $n \geqslant 3$, there is a t_1 that takes the lower bound). It is natural to ask whether the interval from which t_1 is barred can be extended on the right-hand side up to $\mathfrak{M}_r$ for some $r > 1$, taking into account that $\mathfrak{M}_r$ is an increasing function of r, at any rate for $r > 0$, since t_0, t_2 whose mean value is being constructed are unequal.* The answer is in the affirmative.

The simple argument that follows does not cover the whole of the Farey series; we have to do away with the beginning of $\mathfrak{F}_n$. It will be appropriate to leave out all terms of $\mathfrak{F}_n$ less than $\frac{1}{2}$ since $\mathfrak{F}_n$ is symmetrical: any two terms equidistant from $\frac{1}{2}$ are complementary. We may thus state

THEOREM 8. *If $t_0 \geqslant \frac{1}{2}$, and t_0, t_1, t_2 are successive terms in $\mathfrak{F}_n$, then either $t_1 \leqslant \mathfrak{M}_{-1}(t_0, t_2)$, or $t_1 > \mathfrak{M}_3(t_0, t_2)$.*

* Cf. formula (2.2).

The proof can be based on the symmetry of $\mathfrak{F}_n$. First, we note that $1-t_2$, $1-t_1$, $1-t_0$ are successive terms of $\mathfrak{F}_n$. It follows from Theorem 5 that t_1 is excluded from the interval

$$1-\mathfrak{M}_1(1-t_0, 1-t_2) \leqslant x < 1-\mathfrak{M}_{-1}(1-t_0, 1-t_2), \tag{4.3}$$

and, of course, from the interval

$$\mathfrak{M}_{-1}(t_0, t_2) < x \leqslant \mathfrak{M}_1(t_0, t_2). \tag{4.4}$$

Plainly, $$1-\mathfrak{M}_1(1-t_0, 1-t_2) = \mathfrak{M}_1(t_0, t_2),$$

so that the intervals (4.3) and (4.4) cover the interval

$$\mathfrak{M}_{-1}(t_0, t_2) < x < 1-\mathfrak{M}_{-1}(1-t_0, 1-t_2).$$

The only additional remark needed for the proof of Theorem 8 is that

$$\mathfrak{M}_3(t_0, t_2) < 1-\mathfrak{M}_{-1}(1-t_0, 1-t_2).$$

The right-hand side is, by the definition of $\mathfrak{M}_{-1}$,

$$\frac{t_0+t_2-2t_0 t_2}{2-(t_0+t_2)}.$$

If we substitute $S = \frac{1}{2}(t_2+t_0)$, $D = \frac{1}{2}(t_2-t_0)$, the inequality to be proved becomes

$$\{\tfrac{1}{2}(S+D)^3+\tfrac{1}{2}(S-D)^3\}^{\frac{1}{3}} < S+\frac{D^2}{1-S}. \tag{4.5}$$

By hypothesis $S > \frac{1}{2}$, and so

$$\frac{D^2}{1-S} > \frac{D^2}{S}.$$

Hence (4.5) is true if

$$\tfrac{1}{2}(S+D)^3+\tfrac{1}{2}(S-D)^3 < \left(S+\frac{D^2}{S}\right)^3;$$

but that is trivial.

It is of some interest that 3 is the best possible suffix of $\mathfrak{M}$ for which the preceding proof works. To show this, let n be odd and take $t_0 = \frac{1}{2}$. One can easily verify that then $t_1 = \frac{1}{2}(n+1)/n$, $t_2 = \frac{1}{2}(n-1)/(n-2)$. For $n \geqslant 5$ we have $\mathfrak{M}_1(t_0, t_2) < t_1$. Suppose now that

$$\mathfrak{M}_r(t_0, t_2) \leqslant 1-\mathfrak{M}_{-1}(1-t_0, 1-t_2) = \frac{t_0+t_2-2t_0 t_2}{2-(t_0+t_2)}.$$

This gives $$\left\{\frac{1}{2}\frac{1}{2^r}+\frac{1}{2}\frac{1}{2^r}\left(\frac{n-1}{n-2}\right)^r\right\}^{1/r} \leqslant \frac{1}{3-\dfrac{n-1}{n-2}}.$$

Writing
$$1+\alpha = \frac{n-1}{n-2},$$
we have
$$1+(1+\alpha)^r \leqslant \frac{2^{r+1}}{(2-\alpha)^r},$$
i.e.
$$(1-\tfrac{1}{2}\alpha)^r\{1+(1+\alpha)^r\} \leqslant 2,$$
and we obtain
$$\tfrac{1}{4}r(r-3)\alpha^2+o(\alpha^2) \leqslant 0.$$
When α tends to zero, that is, $n \to \infty$, we have, finally,
$$r \leqslant 3.$$

The argument is, of course, not a definite answer to the question whether in Theorem 8 the suffix 3 could be replaced by any greater number, particularly for large n. On the other hand, if a proposition similar to Theorem 8 is to be valid for all $n \geqslant 3$, the example of $\mathfrak{F}_5$ proves that in the case $t_1 \leqslant \mathfrak{M}_{-1}$ the suffix -1 is the best possible, while the best possible $r > 0$ for the alternative $t_1 > \mathfrak{M}_r$ is less than 4.

THE FIRST AND SECOND VARIATIONS OF THE VOLUME INTEGRAL IN RIEMANNIAN SPACE

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LEVI-CIVITA's equations of geodesic deviation (**1**) were obtained by Berwald (**2**) on forming the Euler equations for the integral of the second variation of the length integral between two fixed points. The Euler equations thus formed for the second variation are the Jacobi equations for the original integral. The same method was later used by Schoenberg (**5**), who made use of the Cartesian (Fermi) coordinates which can be defined at every point of a curve in Riemannian space.

The generalization of the equations of geodesic deviation to minimal subspaces was carried out by Bortolotti (**4**). It is the purpose of this note to obtain Bortolotti's equations from the second variation of the 'volume integral'. The method used will be similar to that used by Synge (**6**) in his study of the first and second variations of the length integral.

1. Let the space V_n have the metric tensor $a_{\lambda\mu}$ ($\lambda, \mu, \nu, \rho, \sigma = 1, 2,..., n$) and let V_m ($m < n$), a subspace immersed in it, be defined by the parametric equations

$$x^\lambda = x^\lambda(u^1, u^2,..., u^m) \equiv x^\lambda(u) \tag{1}$$

with the metric tensor b_{ij} ($i, j, k, l, m, n = 1, 2,..., m$) connected with $a_{\lambda\mu}$ by the equations

$$b_{ij} = a_{\lambda\mu}\frac{\partial x^\lambda}{\partial u^i}\frac{\partial x^\mu}{\partial u^j}. \tag{2}$$

We now consider a one-parameter congruence of V_m's defined by

$$x^\lambda = x^\lambda(u^1, u^2,..., u^m, t) \tag{3}$$

with $$x^\lambda(u^1, u^2,..., u^m, 0) \equiv x^\lambda(u^1,..., u^m),$$

and the various members of the congruence for small values of t will be regarded as *varied* positions of the *base* V_m. It is assumed that all the V_m's defined by (3) have a common boundary, so that along that boundary the vector $\partial x^\lambda/\partial t$ vanishes identically.

We shall use the notation

$$\frac{\partial x^\lambda}{\partial u^i} = B_i^\lambda, \qquad \frac{\partial x^\lambda}{\partial t} = v^\lambda, \qquad B_\mu^i = b^{ik} a_{\mu\rho} B_k^\rho, \qquad B_i^\lambda B_\mu^i = B_\mu^\lambda, \tag{4}$$

$$F = \sqrt{b}, \qquad b = |b_{ij}|, \qquad F_\lambda | = \frac{\partial F}{\partial x^\lambda}, \qquad F |_\lambda^i = \frac{\partial F}{\partial B_i^\lambda},$$

$$F_\lambda |_\mu^i = \frac{\partial}{\partial B_i^\mu}(F_\lambda |) = \frac{\partial}{\partial x^\lambda}(F |_\mu^i), \qquad F |_{\lambda\mu}^{ij} = \frac{\partial^2 F}{\partial B_i^\lambda \partial B_j^\mu}.$$

We now proceed to calculate these quantities. We have

$$F_\lambda | = \tfrac{1}{2}\sqrt{b}\, b^{ij} B_{ij}^{\mu\nu} \frac{\partial a_{\mu\nu}}{\partial x^\lambda},$$

which, on using the expression for $\partial a_{\mu\nu}/\partial x^\lambda$ in terms of the Christoffel three-index symbols of the second kind, and simplifying, becomes

$$F_\lambda | = F B_\sigma^\rho \left\{ {\sigma \atop \rho\lambda} \right\}. \tag{5}$$

Similarly,

$$F |_\lambda^i = \tfrac{1}{2}\sqrt{b}\, b^{kl} \frac{\partial}{\partial B_i^\lambda} b_{kl},$$

which, on using the expression (2) for b_{kl}, becomes

$$F |_\lambda^i = \sqrt{b}\, b^{ik} a_{\lambda\sigma} B_k^\sigma = F B_\lambda^i. \tag{6}$$

Again, on writing $C_\mu^\sigma = \delta_\mu^\sigma - B_\mu^\sigma = \delta_\mu^\sigma - B_i^\sigma B_\mu^i$, we have

$$F_\lambda |_\mu^j = F B_k^\rho \frac{\partial a_{\rho\sigma}}{\partial x^\lambda} (\tfrac{1}{2} b^{ik} B_i^\sigma B_\mu^j + b^{jk} C_\mu^\sigma)$$

and, finally, $\quad F |_{\lambda\mu}^{ij} = (F |_\lambda^i) |_\mu^j = F |_\mu^j B_\lambda^i + F(B_\lambda^i |_\mu^j),$

so that we have to evaluate $B_\lambda^i |_\mu^j \equiv \dfrac{\partial B_\lambda^i}{\partial B_j^\mu}$, where $B_\lambda^i = b^{ik} a_{\lambda\sigma} B_k^\sigma$. The first stage of the calculation gives us

$$B_\lambda^i |_\mu^j = b^{ij} a_{\lambda\mu} + a_{\lambda\sigma} B_k^\sigma \frac{\partial b^{ik}}{\partial B_j^\mu}.$$

On using (2) and the relation $b^{ik} b_{kj} = \delta_j^i$, we get

$$\frac{\partial b^{ik}}{\partial B_j^\mu} = -b^{jk} B_\mu^i - b^{ij} B_\mu^k,$$

so that

$$B_\lambda^i |_\mu^j = b^{ij} a_{\lambda\mu} - B_{\lambda\mu}^{ji} - b^{ij} a_{\lambda\rho} B_\mu^\rho. \tag{7}$$

On inserting this value we then get

$$F |_{\lambda\mu}^{ij} = F(b^{ij} a_{\lambda\mu} - b^{ij} a_{\lambda\rho} B_\mu^\rho + B_{\lambda\mu}^{ij} - B_{\mu\lambda}^{ij}). \tag{8}$$

2. We now consider the integral

$$L(t) = \int_\Sigma F(u^1,\ldots,u^m,t)\, du^1 du^2 \ldots du^m \tag{9}$$

taken over a region Σ of variation of the u's, which, when $t = 0$, reduces to

$$L(0) = \int_{\Sigma} \sqrt{b}\, du^1 ... du^m,$$

the volume of the closed region Σ of the space V_m bounded by a fixed boundary γ. We now assume that the function F and its first two derivatives with respect to t are continuous and bounded functions of the $m+1$ variables in the interior points of the region Σ and for values of t in the immediate neighbourhood of $t = 0$. We can therefore apply Leibniz's rule for differentiation under the integral sign, and the first few terms of the Taylor expansion of $L(t)$ as a function of t will be

$$L(t) = L(0) + tL'(0) + \frac{t^2}{2!} L''(0) + ...,$$

where

$$L'(0) = \int_{\Sigma} F'(0)\, du^1 ... du^m; \qquad L''(0) = \int_{\Sigma} F''(0)\, du^1 du^2 ... du^m. \tag{10}$$

We now proceed to calculate $F'(0)$ and $F''(0)$.

We have
$$\frac{dF}{dt} = F_\lambda \,|\, \frac{\partial x^\lambda}{\partial t} + F\,|^i_\lambda \frac{\partial}{\partial t}(B^\lambda_i),$$

so that, using (4) and the fact that $\frac{\partial}{\partial t}(B^\lambda_i) = \frac{\partial v^\lambda}{\partial u^i} = \partial_i v^\lambda$, we shall have

$$F'(0) = F_\lambda \,|\, v^\lambda + F\,|^i_\lambda \partial_i v^\lambda. \tag{11}$$

On using (5) and (6) the right-hand side becomes

$$F[B^\rho_\sigma \{^{\sigma}_{\rho\lambda}\} v^\lambda + B^i_\lambda \partial_i v^\lambda] = F B^i_\lambda D_i v^\lambda,$$

where
$$D_i v^\lambda = \partial_i v^\lambda + \{^{\lambda}_{\mu\nu}\} v^\mu B^\nu_i.$$

We therefore write
$$F'(0) = F\,|^i_\lambda D_i v^\lambda. \tag{12}$$

We then proceed to the calculation of $F''(0)$. We have, on assuming that the value $t = 0$ is always substituted,

$$\frac{d^2F}{dt^2} = \frac{d}{dt}(F\,|^i_\lambda D_i v^\lambda) = \left(\frac{\partial F\,|^i_\lambda}{\partial B^\mu_j}\frac{\partial B^\mu_j}{\partial t} + \frac{\partial F\,|^i_\lambda}{\partial x^\mu}\frac{\partial x^\mu}{\partial t}\right) D_i v^\lambda + F\,|^i_\lambda \frac{\partial}{\partial t}(D_i v^\lambda)$$

$$= (F\,|^{ij}_{\lambda\mu} \partial_j v^\mu + F_\mu \,|^i_\lambda v^\mu) D_i v^\lambda + F\,|^i_\lambda \frac{\partial}{\partial t}(D_i v^\lambda),$$

and this may be written in the form

$$F\,|^{ij}_{\lambda\mu} D_i v^\lambda D_j v^\mu + F\,|^i_\lambda \frac{D}{\partial t}(D_i v^\lambda) + \Phi^i_{\lambda\mu} D_i v^\lambda v^\mu,$$

where the $D_i v^\lambda$ is treated as a vector of V_n and consequently the index i is merely ordinal, so that

$$\frac{D}{\partial t}(D_i v^\lambda) = \frac{\partial}{\partial t}(D_i v^\lambda) + \{^{\lambda}_{\mu\nu}\} D_i v^\mu \frac{\partial x^\nu}{\partial t} = \frac{\partial}{\partial t}(D_i v^\lambda) + \{^{\lambda}_{\mu\nu}\} D_i v^\mu v^\nu.$$

The $\Phi^i_{\lambda\mu}$ denote covariant tensors of the second order in the indices λ and μ, which evidently vanish when the coordinates used are the normal coordinates at the point, and which therefore vanish for all coordinate systems, so that

$$F''(0) = F \,|^{ij}_{\lambda\mu} D_i v^\lambda D_j v^\mu + F \,|^{i}_{\lambda} \frac{D}{\partial t}(D_i v^\lambda). \tag{13}$$

If we now introduce the notations

$$a_{\lambda\mu} B^\lambda_i D_j v^\mu = \omega_{ij}, \qquad b^{ij}\omega_{ij} = \omega, \qquad a_{\lambda\mu} D_i v^\lambda D_j v^\mu = \pi_{ij},$$

the equations (12) and (13) become, on using (6) and (8),

$$F'(0) = \sqrt{b}\,\omega \tag{14}$$

and

$$F''(0) = \sqrt{b}\left[b^{ij}\pi_{ij} + \omega^2 - 2\omega^{ij}\omega_{ij} + B^i_\lambda \frac{D}{\partial t}(D_i v^\lambda)\right]. \tag{15}$$

It will be convenient later to have the last term inside the bracket modified in accordance with the relation

$$\frac{D}{\partial t}(D_i v^\lambda) - D_i\left(\frac{D}{\partial t} v^\lambda\right) = R_{\sigma\rho,\mu}^{\cdots\lambda} B^\rho_i v^\sigma v^\mu,$$

where the particular convention with regard to $R_{\sigma\rho,\mu}^{\cdots\lambda}$ is that of Schouten,* so that (15) becomes

$$F''(0) = \sqrt{b}\left[b^{ij}\pi_{ij} + \omega^2 - 2\omega^{ij}\omega_{ij} + B^i_\lambda D_i\left(\frac{D}{\partial t} v^\lambda\right) + R_{\nu\rho,\mu}^{\cdots\lambda} B^\rho_\lambda v^\mu v^\lambda\right]. \tag{16}$$

Passing now to the integral itself, we write $d\sigma = \sqrt{b}\,du^1 du^2 \dots du^m$.

Then

$$L(0) = \int_\Sigma d\sigma, \qquad L'(0) = \int_\Sigma \omega\, d\sigma. \tag{17}$$

We can now state theorems similar to those stated by Synge (**6**) for curves, on considering the conditions for the vanishing of ω, assuming a fixed boundary.

I. *The first variation vanishes when the variation vector v^λ is propagated parallelly on the surface.*

II. *The first variation vanishes if the covariant derivative of v^λ is normal to V_m at every point.*

* (7) I, 110.

Since we are dealing with a fixed boundary for which $v^\lambda = 0$, the right-hand side of (17) can be modified by partial integration to

$$-\int_\Sigma (D_i B^i_\lambda) v^\lambda \, d\sigma = -\int_\Sigma M_\lambda v^\lambda \, d\sigma,$$

where M_λ are the covariant components of the mean-curvature-normal vector.* Hence we may state further

III. *The first variation vanishes if the variation vector is normal to the mean-curvature-normal vector at all points of* V_m.

IV. *The first variation vanishes if* V_m *is minimal, so that* $M^\lambda = 0$ *at all points of* V_m.

3. Returning now to the expression (16) for $F''(0)$, we can further modify the term $B^i_\lambda D_i\left(\frac{D}{\partial t} v^\lambda\right)$ by integrating by parts and using the fact that v^λ vanishes identically on the boundary, so that $\frac{D}{\partial t} v^\lambda$ also vanishes on the boundary. This gives

$$\int_\Sigma B^i_\lambda D_i\left(\frac{D}{\partial t} v^\lambda\right) d\sigma = -\int_\Sigma M_\lambda \frac{D}{\partial t} v^\lambda \, d\sigma,$$

so that $L''(0)$ becomes

$$L''(0) = \int_\Sigma \left[(b^{ij}\pi_{ij} - \omega^{ij}\omega_{ij}) + (\omega^2 - \omega^{ij}\omega_{ij}) - M_\lambda \frac{D}{\partial t} v^\lambda + R_{\nu\rho,\mu}^{\cdot\cdot\cdot\lambda} B^\rho_\lambda v^\mu v^\nu\right] d\sigma, \tag{18}$$

where the grouping $b^{ij}\pi_{ij} - \omega^{ij}\omega_{ij}$ has a simple geometrical interpretation.† It is equal to $\dfrac{\sin^2\begin{pmatrix} \alpha \\ m, g \end{pmatrix}}{dt^2}$, where $\begin{pmatrix} \alpha \\ m, g \end{pmatrix}$ is the 'plane angle' between the m-plane determined by the tangent vectors B^λ_i to V_m and the *varied* m-plane determined by the vectors $'B^\lambda_i \equiv B^\lambda_i + \partial_i v^\lambda \, dt$.

For the particular case in which the variation vector is normal to the V_m, so that $v^\lambda = C^\lambda_p w^p$ $(p, q, r = m+1, \ldots, n)$, the bracket $(b^{ij}\pi_{ij} - \omega^{ij}\omega_{ij})$ reduces to a single term‡

$$\sum_{p=m+1}^{n} b^{ij} D_i w^p D_j w^p = \Delta_1(w^p, w_p)$$

to which Bortolotti gives the name *normal curvature associated with the congruence* w^p.

* (7) II, 87. † (8) 293. ‡ (8) 294.

In this case ω_{ij} becomes $a_{\lambda\mu}\, B_i^\lambda\, D_j\, C_p^\mu\, w^p$, and, using $a_{\lambda\mu}\, B_i^\lambda\, C_p^\mu = 0$, we have

$$a_{\lambda\mu}\, B_i^\lambda\, D_j\, C_p^\mu = -a_{\lambda\mu}\, C_p^\mu\, D_j\, B_i^\lambda = -\overset{p}{h}_{ij}$$

with $\overset{p}{h}_{ij}$ as defined by Schouten.* Accordingly,

$$\omega_{ij} = -\overset{p}{h}_{ij}\, w_p = -\underset{p}{h}_{ij}\, w^p, \qquad \omega = M^p w_p = M_p\, w^p = M,$$

and

$$\omega^2 - \omega^{ij}\omega_{ij} = b^{im}b^{jn}\Big(\underset{p}{h}_{im}\,\underset{q}{h}_{jn} - \underset{p}{h}_{mn}\,\underset{q}{h}_{ij}\Big)w^p w^q = \Big(M_p\, M_q - \underset{p}{h}{}^{ij}\underset{q}{h}_{ij}\Big)w^p w^q.$$

Similarly, the last term may be modified to $R_{\nu\rho,\mu\lambda}\, B_{ij}^{\rho\lambda}\, b^{ij} C_p^\mu\, C_q^\nu\, w^p w^q$. Hence, without any assumption about the first variation, the integral of the second variation for a normal variation is

$$L''(0) = \int\limits_{\Sigma} \Big[\Delta_1(w^p, w_p) + M^2 - \underset{p}{h}{}^{ij}\underset{q}{h}_{ij}\, w^p w^q - M_\lambda \frac{D}{\partial t} v^\lambda + {} + R_{\nu\rho,\mu\lambda}\, B_{ij}^{\rho\lambda}\, b^{ij} C_p^\mu\, C_q^\nu\, w^p w^q\Big]\, d\sigma, \tag{19}$$

which, for a minimal subspace V_m, becomes simply

$$L''(0) = \int\limits_{\Sigma} [\Delta_1(w^p, w_p) - U_{pq}\, w^p w^q]\, d\sigma, \tag{20}$$

where

$$U_{pq} = \underset{p}{h}{}^{ij}\underset{q}{h}_{ij} - R_{\nu\rho,\mu\lambda}\, B_{ij}^{\rho\lambda}\, b^{ij} C_{pq}^{\mu\nu}.$$

The integral now represents a function of w and its derivative with respect to u which can be used as the integrand in an ordinary problem of the calculus of variations. Let us write

$$H\Big(w, \frac{\partial w}{\partial u}\Big) = \sqrt{b}[\Delta_1(w^p, w_p) - U_{pq}\, w^p w^q]. \tag{21}$$

Then, forming the Euler equations

$$\frac{\partial}{\partial u^i}\Big(\frac{\partial H}{\partial(\partial_i\, w^p)}\Big) - \frac{\partial H}{\partial w^p} = 0, \tag{22}$$

we shall have

$$b^{ij} D_i\, D_j\, w^p + U_{pq}\, w^q = 0, \tag{23}$$

and this is equivalent to Bortolotti's equation† to be satisfied by the orthogonal components of the variation vector in order that the new submanifold shall also be a minimal subspace of the V_n,† the different sign in the R term being due to the different convention with regard to the position of the indices.

Now let us consider the particular case of a minimal V_{n-1} in V_n,

* (7) II, 82. † (4) equation 137.

in which case $v^\lambda = l^\lambda w$, where l^λ is the unique unit normal vector. In that case $D_i w^p$ becomes simply $\partial_i w$, and the expression for the second variation (20) becomes

$$L''(0) = \int_\Sigma (b^{ij}\partial_i w \partial_j w - Uw^2)\, d\sigma, \tag{24}$$

where $$U = U_{nn} = h^{ij}h_{ij} - R_{\nu\rho\mu\lambda} B^{\rho\lambda}_{ij} b^{ij} l^\mu l^\nu$$

is an invariant which may be expressed in terms of the relative curvature K_r of V_{n-1} in V_n and of the 'Ricci mean curvature' as follows:*

$$h^{ij}h_{ij} = -(n-1)(n-2)K_r,$$

$$R_{\nu\rho\mu\lambda} B^{\rho\lambda}_{ij} b^{ij} l^\mu l^\nu = -M_l,$$

so that $$U = M_l - (n-1)(n-2)K_r. \tag{25}$$

This invariant U is of interest as the 'invariant of Koschmieder' which appears in the expression given by Berwald for the second variation of the volume integral in metric spaces based on the notion of area.† On repeating the operations corresponding to the steps (21)–(23) we shall have

$$b^{ij} D_i D_j w + Uw = 0, \quad \text{i.e. } \Delta_2 w + Uw = 0, \tag{26}$$

which coincides with equations (114) of Bortolotti.

* (4) equations 103, 109, 112, 113.

† (3) equation 33.2.

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